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# ON $L_p$ -SOLVABILITY OF STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

ISTVÁN GYÖNGY AND SIZHOU WU

ABSTRACT. A class of (possibly) degenerate stochastic integro-differential equations of parabolic type is considered, which includes the Zakai equation in nonlinear filtering for jump diffusions. Existence and uniqueness of the solutions are established in Bessel potential spaces.

## 1. INTRODUCTION

We consider the equation

$$\begin{aligned} du_t(x) &= (\mathcal{A}_t u_t(x) + f_t(x)) dt + (\mathcal{M}_t^r u_t(x) + g_t^r(x)) dw_t^r \\ &+ \int_Z (u_{t-}(x + \eta_{t,z}(x)) - u_{t-}(x) + \gamma_{t,z}(x)u_{t-}(x + \eta_{t,z}(x)) + h_t(x, z)) \tilde{\pi}(dz, dt) \end{aligned} \quad (1.1)$$

for  $(t, x) \in [0, T] \times \mathbb{R}^d := H_T$  with initial condition

$$u_0(x) = \psi(x) \quad \text{for } x \in \mathbb{R}^d, \quad (1.2)$$

on a filtered probability space  $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ , carrying a sequence  $w = (w_t^i)_{i=1}^\infty$  of independent  $\mathcal{F}_t$ -Wiener processes and an  $\mathcal{F}_t$ -Poisson martingale measure  $\tilde{\pi}(dz, dt) = \pi(dz, dt) - \mu(dz) \otimes dt$ , where  $\pi(dz, dt)$  is an  $\mathcal{F}_t$ -Poisson random measure with a  $\sigma$ -finite characteristic measure  $\mu(dz)$  on a measurable space  $(Z, \mathcal{Z})$  with a countably generated  $\sigma$ -algebra  $\mathcal{Z}$ . We note that here, and later on, the summation convention is used with respect to repeated (integer-valued) indices and multi-numbers.

In the above equation  $\mathcal{A}_t$  is an integro-differential operator of the form  $\mathcal{A}_t = \mathcal{L}_t + \mathcal{N}_t^\xi + \mathcal{N}_t^\eta + \mathcal{R}_t$  with a “zero-order” linear operator  $\mathcal{R}_t$  specified later, a second order differential operator

$$\mathcal{L}_t = a_t^{ij}(x)D_{ij} + b_t^i(x)D_i + c_t(x),$$

and integral operators  $\mathcal{N}_t^\xi$  and  $\mathcal{N}_t^\eta$  defined by

$$\begin{aligned} \mathcal{N}_t^\xi \varphi(x) &= \int_Z \varphi(x + \xi_{t,z}(x)) - \varphi(x) - \xi_{t,z}(x) \nabla \varphi(x) + \lambda_{t,z}^\xi(x) (\varphi(x + \xi_{t,z}(x)) - \varphi(x)) \nu(dz), \\ \mathcal{N}_t^\eta \varphi(x) &= \int_Z \varphi(x + \eta_{t,z}(x)) - \varphi(x) - \eta_{t,z}(x) \nabla \varphi(x) + \lambda_{t,z}^\eta(x) (\varphi(x + \eta_{t,z}(x)) - \varphi(x)) \mu(dz) \end{aligned} \quad (1.3)$$

for a suitable class of real-valued functions  $\varphi$  on  $\mathbb{R}^d$  for each  $t \in [0, T]$ , where  $\nu(dz)$  is a fixed  $\sigma$ -finite measure on  $(Z, \mathcal{Z})$ . The coefficients  $a^{ij}$ ,  $b^i$  and  $c$  are real functions on  $\Omega \times H_T$  for

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$i, j = 1, 2, \dots, d$ , and  $\lambda^\xi$ ,  $\lambda^\eta$  and  $\gamma$  are real functions on  $\Omega \times H_T \times Z$ , and  $\eta = (\eta_{t,z}^i(x))$  and  $\xi = (\xi_{t,z}^i(x))$  are  $\mathbb{R}^d$ -valued functions of  $(\omega, t, x, z) \in \Omega \times H_T \times Z$ . Under “zero-order operators” we mean bounded linear operators  $\mathcal{R}$  mapping the Sobolev spaces  $W_p^k$  into themselves for some  $k \geq 0$ . For each integer  $r \geq 1$  the operator  $\mathcal{M}_t^r$  is a first order differential operator of the form

$$\mathcal{M}_t^r = \sigma_t^{ir}(x)D_i + \beta_t^r(x).$$

The coefficients  $\sigma^{ir}$  and  $\beta^r$  are real functions on  $\Omega \times H_T$  for  $i, j = 1, 2, \dots, d$  and integers  $r \geq 1$ . The free terms  $f$  and  $g^r$  are real functions defined on  $\Omega \times H_T$  for every  $r \geq 1$ , and  $h$  is a real function defined on  $\Omega \times H_T \times Z$ . The stochastic differentials in equation (1.1) are understood in Itô’s sense, see the definition of a solution in the next section.

We are interested in the solvability of the above problem in  $L_p$ -spaces. We note that equation (1.1) may degenerate, i.e., the pair of linear operators  $(\mathcal{L}, \mathcal{M})$  satisfies only the *stochastic parabolicity* condition, Assumption 2.1 below, and the operators  $\mathcal{N}^\xi$  and  $\mathcal{N}^\eta$  may also degenerate. Our main result, Theorem 2.1 states that under the stochastic parabolicity condition on the operators  $(\mathcal{L}, \mathcal{M})$ ,  $\mathcal{N}^\xi$ ,  $\mathcal{N}^\eta$ , and appropriate regularity conditions on their coefficients and on the initial and free data, the Cauchy problem (1.1)-(1.2) has a unique generalised solution  $u = (u_t)_{t \in [0, T]}$  for any given  $T$ . Moreover, this theorem describes the temporal and spatial regularity of  $u$  in terms of Bessel potential spaces  $H_p^n$ , and presents also a supremum estimate in time. The uniqueness of the solution is proved by an application of a theorem on Itô’s formula from [17], which generalises a theorem of Krylov in [24] to the case of jump processes. The existence of a generalised solution is proved in several steps. First we obtain a priori estimates in Sobolev spaces  $W_p^n$  for integers  $n \in [0, m]$ , where  $m$  is a parameter measuring the spatial smoothness of the coefficients and the data in (1.1)-(1.2). These estimates allow us to construct a generalised solution by approximating (1.1)-(1.2) with non-degenerate equations with smooth coefficients and compactly supported smooth data in  $x \in \mathbb{R}^d$ . Thus we see that a solution operator, mapping the initial and free data into the solution of (1.1)-(1.2), exists and it is a bounded linear operator in appropriate  $L_p$ -spaces. Hence by interpolation we get our a priori estimates in Bessel potential spaces  $H_p^n$  for any given  $p \geq 2$  and real number  $n \in [0, m]$ . We obtain essential supremum estimates in time for the solution from integral estimates, by using the simple fact that the essential supremum of Lebesgue functions over an interval  $[0, T]$  is the limit of their  $L_r([0, T])$ -norm as  $r \rightarrow \infty$ . Hence we get the temporal regularity of the solution formulated in our main theorem by using Theorem 2.2 on Itô’s formula in [17], an extension of Lemma 5.3 in [9] and a well-known interpolation inequality, Theorem 4.1(v) below.

In the literature there are many results on stochastic integral equations with unbounded operators, driven by jump processes and martingale measures. A general existence and uniqueness theorem for stochastic evolution equations with nonlinear operators satisfying stochastic coercivity and monotonicity conditions is proved in [15], which generalises some results in [25] and [32] to stochastic evolution equations driven by semimartingales and random measures. Further generalisations are obtained and the asymptotic behaviour of the solutions are investigated in [28]. For a monograph on stochastic evolution equations driven by Lévy noise we refer to [33].

The main theorem in [15] implies the existence of a unique generalised solution to (1.1)-(1.2) in  $L_2$ -spaces when instead of the stochastic parabolicity condition (2.1) in Assumption

2.1 below, the strong stochastic parabolicity condition,

$$\sum_{i,j=1}^d \sum_{r=1}^{\infty} (a^{ij} - \frac{1}{2} b^{ir} b^{jr}) z^i z^j \geq \lambda \sum_{i=1}^d |z^i|^2 \quad \text{for all } z = (z^1, z^2, \dots, z^d) \in \mathbb{R}^d$$

with a constant  $\lambda > 0$  is assumed on  $(\mathcal{L}, \mathcal{M})$ . Under the weaker condition of stochastic parabolicity the solvability of (1.1)-(1.2) in  $L_2$ -spaces is investigated and existence and uniqueness theorems are presented in [8] and [27]. The first result on solvability in  $L_p$ -spaces for the stochastic PDE problem (1.1)-(1.2) with  $\xi = \eta = 0$  and  $h = 0$  was obtained in [26], and was improved in [16]. However, there is a gap in the proof of the crucial a priori estimate in [26]. This gap is filled in and more general results on solvability in  $L_p$ -spaces for systems of stochastic PDEs driven by Wiener processes are presented in [12]. As far as we know Theorem 2.1 below is the first result on solvability in  $L_p$ -spaces of stochastic integro-differential equations (SIDEs) without any non-degeneracy conditions. It generalises the main result of [9] on deterministic integro-differential equations to SIDEs. Our motivation to study equation (1.1) comes from nonlinear filtering of jump-diffusion processes, and we want to apply Theorem 2.1 to filtering problems in a continuation of the present paper.

We note that under non-degeneracy conditions SIDEs have been investigated with various generalities in the literature, and very nice results on their solvability in  $L_p$ -spaces have recently been obtained. In particular,  $L_p$ -theories for such equations have been developed in [20], [21], [29], [30] and [31], which extend some results of the  $L_p$ -theory of Krylov [22] to certain classes of equations with nonlocal operators. See also [7] for an  $L_p$ -theory for stochastic PDEs driven by Lévy processes, [36] for an existence and uniqueness theorem for stochastic quasi-geostrophic equations driven by Poisson martingale measures, and [6], [10] and [35] for  $L_p$  theory of deterministic equations with nonlocal operators. Nonlinear filtering problems and the related equations describing the conditional distributions have been extensively studied in the literature. For results in the case of jump-diffusion models see, for example, [2], [4], [11] and [14].

In conclusion, we introduce some notions and notations used throughout this paper. All random elements are given on the filtered probability space  $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ . We assume that  $\mathcal{F}$  is  $P$ -complete, the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is right-continuous, and  $\mathcal{F}_0$  contains all  $P$ -zero sets of  $\mathcal{F}$ . The  $\sigma$ -algebra of the predictable subsets of  $\Omega \times [0, \infty)$  is denoted by  $\mathcal{P}$ . For notations, notions and results concerning Lévy processes, Poisson random measures and stochastic integrals we refer to [1], [3] and [19].

For vectors  $v = (v^i)$  and  $w = (w^i)$  in  $\mathbb{R}^d$  we use the notation  $vw = \sum_{i=1}^m v^i w^i$  and  $|v|^2 = \sum_i |v^i|^2$ . For real-valued Lebesgue measurable functions  $f$  and  $g$  defined on  $\mathbb{R}^d$  the notation  $(f, g)$  means the integral of the product  $fg$  over  $\mathbb{R}^d$  with respect to the Lebesgue measure on  $\mathbb{R}^d$ . A finite list  $\alpha = \alpha_1 \alpha_2, \dots, \alpha_n$  of numbers  $\alpha_i \in \{1, 2, \dots, d\}$  is called a multi-number of length  $|\alpha| := n$ , and the notation

$$D_\alpha := D_{\alpha_1} D_{\alpha_2} \dots D_{\alpha_n}$$

is used for integers  $n \geq 1$ , where

$$D_i v = \frac{\partial}{\partial x^i} v$$

is the generalised derivative of a real-valued function  $v$  with respect to  $x_i$ ,  $i = 1, 2, \dots, d$ . We use also the multi-number  $\epsilon$  of length 0 such that  $D_\epsilon$  means the identity operator. For

an integer  $n \geq 0$  and functions  $v$  on  $\mathbb{R}^d$ , whose generalised derivatives up to order  $n$  are functions, we use the notation  $D^n v$  for the collection  $\{D_\alpha v : |\alpha| = n\}$ , and define

$$|D^n v|^2 = \sum_{|\alpha|=n} |D_\alpha v|^2.$$

For differentiable functions  $v = (v^1, \dots, v^d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  the notation  $Dv$  means the Jacobian matrix whose  $j$ -th entry in the  $i$ -th row is  $D_j v^i$ . When we talk about “derivatives up to order  $n$ ” of a function for some nonnegative integer  $n$ , then we always include the zero-order derivative, i.e. the function itself. The space of smooth functions  $\varphi = \varphi(x)$  with compact support on the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$  is denoted by  $C_0^\infty$ .

For  $p, q \geq 1$  we denote by  $\mathcal{L}_q = \mathcal{L}_q(Z, \mathbb{R})$  the Banach spaces of  $\mathbb{R}$ -valued  $\mathcal{Z}$ -measurable functions  $h = h(z)$  of  $z \in Z$  such that

$$|h|_{\mathcal{L}_q}^q = \int_Z |h(z)|^q \mu(dz) < \infty.$$

The notation  $\mathcal{L}_{p,q}$  means the space  $\mathcal{L}_p \cap \mathcal{L}_q$  with the norm

$$|v|_{\mathcal{L}_{p,q}} = \max(|v|_{\mathcal{L}_p}, |v|_{\mathcal{L}_q}) \quad \text{for } v \in \mathcal{L}_p \cap \mathcal{L}_q.$$

The space of sequences  $\nu = (\nu^1, \nu^2, \dots)$  of real numbers  $\nu^k$  with finite norm

$$|\nu|_{l_2} = \left( \sum_{k=1}^{\infty} |\nu^k|^2 \right)^{1/2}$$

is denoted by  $l_2$ .

The Borel  $\sigma$ -algebra of a separable Banach space  $V$  is denoted by  $\mathcal{B}(V)$ , and for  $p \geq 0$  the notations  $L_p([0, T], V)$  and  $L_p(\mathbb{R}^d, V)$  are used for the space of  $V$ -valued Borel-measurable functions  $f$  on  $[0, T]$  and  $g$  on  $\mathbb{R}^d$ , respectively, such that  $|f|_V^p$  and  $|g|_V^p$  have finite Lebesgue integral over  $[0, T]$  and  $\mathbb{R}^d$ , respectively. For  $p \geq 1$  and  $f \in L_p(\mathbb{R}^d, V)$  we use the notation  $|f|_{L_p}$ , defined by

$$|f|_{L_p}^p = \int_{\mathbb{R}^d} |f(x)|_V^p dx < \infty.$$

In the sequel,  $V$  will be  $\mathbb{R}$ ,  $l_2$  or  $\mathcal{L}_{p,q}$ . For integer  $n \geq 0$  the space of functions from  $L_p(\mathbb{R}^d, V)$ , whose generalised derivatives up to order  $n$  are also in  $L_p(\mathbb{R}^d, V)$ , is denoted by  $W_p^n = W_p^n(\mathbb{R}^d, V)$  with the norm

$$|f|_{W_p^n} := \sum_{|\alpha| \leq n} |D_\alpha f|_{L_p}.$$

By definition  $W_p^0(\mathbb{R}^d, V) = L_p(\mathbb{R}^d, V)$ .

For  $m \in \mathbb{R}$  and  $p \in (1, \infty)$  we use the notation  $H_p^m = H_p^m(\mathbb{R}^d; V)$  for the Bessel potential space with exponent  $p$  and order  $m$ , defined as the space of  $V$ -valued generalised functions  $\varphi$  on  $\mathbb{R}^d$  such that

$$(1 - \Delta)^{m/2} \varphi \in L_p \quad \text{and} \quad |\varphi|_{H_p^m} := |(1 - \Delta)^{m/2} \varphi|_{L_p} < \infty,$$

where  $\Delta = \sum_{i=1}^d D_i^2$ . Moreover, we use  $\mathbb{H}_p^m$  to denote the space of  $\mathcal{P}$ -measurable functions from  $\Omega \times [0, T]$  to  $H_p^m$  such that

$$|f|_{\mathbb{H}_p^m}^p := E \int_0^T |f_t|_{H_p^m}^p dt < \infty.$$

We will often omit the target space  $V$  in the notations  $W_p^n(V)$ ,  $H_p^m(V)$ , and  $\mathbb{H}_p^m(V)$  for convenience if  $V = \mathbb{R}$ , and we use  $\mathbb{L}_p$  to denote  $\mathbb{H}_p^0$ . When  $V = \mathcal{L}_{p,q}$  we use  $W_{p,q}^n$ ,  $H_{p,q}^m$  and  $\mathbb{H}_{p,q}^m$  to denote  $W_p^n(\mathcal{L}_{p,q})$ ,  $H_p^m(\mathcal{L}_{p,q})$  and  $\mathbb{H}_p^m(\mathcal{L}_{p,q})$  respectively, and we use  $\mathbb{L}_{p,q}$  to denote  $\mathbb{H}_{p,q}^0$ .

*Remark 1.1.* If  $V$  is a UMD space, see for example [18] for the definition of UMD spaces, then by Theorem 5.6.11 in [18] for  $p > 1$  and integers  $n \geq 1$  we have  $W_p^n(V) = H_p^n(V)$  with equivalent norms. Clearly,  $\mathcal{L}_{p,q}$  is a UMD space for  $p, q \in (1, \infty)$ , which implies  $W_{p,q}^n = H_{p,q}^n$  for non-negative integers  $n$  and  $p, q \in (1, \infty)$ .

## 2. FORMULATION OF THE RESULTS

To formulate our assumptions we fix a constant  $K$ , a non-negative real number  $m$ , an exponent  $p \in [2, \infty)$ , and non-negative  $\mathcal{Z}$ -measurable real-valued functions  $\bar{\eta}$  and  $\bar{\xi}$  on  $Z$  such that they are bounded by  $K$  and

$$K_{\bar{\eta}}^2 := \int_Z |\bar{\eta}(z)|^2 \mu(dz) < \infty \quad K_{\bar{\xi}}^2 := \int_Z |\bar{\xi}(z)|^2 \nu(dz) < \infty.$$

We denote by  $\lceil m \rceil$  the smallest integer which is greater than or equal to  $m$ , and  $\lfloor m \rfloor$  the largest integer which is less than or equal to  $m$ .

**Assumption 2.1.** The derivatives in  $x \in \mathbb{R}^d$  of  $a^{ij}$ ,  $b^i$  and  $c$  up to order  $\max\{\lceil m \rceil, 2\}$ ,  $\max\{\lceil m \rceil, 1\}$  and  $\lceil m \rceil$ , respectively, are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions on  $\Omega \times H_T$ , bounded by  $K$  for all  $i, j = 1, 2, \dots, d$ . The functions  $\sigma^i = (\sigma^{ir})_{r=1}^\infty$  and  $\beta = (\beta^r)_{r=1}^\infty$  and their derivatives up to order  $\lceil m \rceil + 1$  are  $l_2$ -valued  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions, bounded by  $K$ . Moreover  $a^{ij} = a^{ji}$  for all  $i, j = 1, \dots, d$ , and for  $P \otimes dt \otimes dx$ -almost all  $(\omega, t, x) \in \Omega \times H_T$

$$\tilde{a}_t^{ij}(x) z^i z^j \geq 0 \quad \text{for all } z = (z^1, \dots, z^d) \in \mathbb{R}^d, \quad (2.1)$$

where

$$\tilde{a}^{ij} = 2a^{ij} - \sigma^{ir} \sigma^{jr}.$$

**Assumption 2.2.** The mapping  $\xi = (\xi^i)$  is an  $\mathbb{R}^d$ -valued  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$ -measurable function on  $\Omega \times [0, T] \times \mathbb{R}^d \times Z$ . Its derivatives in  $x \in \mathbb{R}^d$  up to order  $\max\{\lceil m \rceil, 3\}$  exist and are continuous in  $x \in \mathbb{R}^d$  such that

$$|D^k \xi| \leq \bar{\xi} \quad k = 0, 1, 2, \dots, \max\{\lceil m \rceil, 3\} := \bar{m}$$

for all  $(\omega, t, x, z) \in \Omega \times H_T \times Z$ . Moreover,

$$K^{-1} \leq |\det(\mathbf{I} + \theta D\xi_{t,z}(x))|$$

for all  $(\omega, t, x, z, \theta) \in \Omega \times H_T \times Z \times [0, 1]$ , where  $\mathbf{I}$  is the  $d \times d$  identity matrix, and  $D\xi$  denotes the Jacobian matrix of  $\xi$  in  $x \in \mathbb{R}^d$ .

**Assumption 2.3.** The function  $\eta = (\eta^i)$  maps  $\Omega \times [0, T] \times \mathbb{R}^d \times Z$  into  $\mathbb{R}^d$  such that Assumption 2.2 holds with  $\eta$  and  $\bar{\eta}$  in place of  $\xi$  and  $\bar{\xi}$ , respectively.

*Remark 2.1.* Assumptions 2.2 and 2.3 imply that for each  $(\omega, t, z, \theta) \in \Omega \times [0, T] \times Z \times [0, 1]$  the mappings

$$\tau_{\theta\xi}(x) := x + \theta\xi_{t,z}(x) \quad \text{and} \quad \tau_{\theta\eta}(x) := x + \theta\eta_{t,z}(x) \quad (2.2)$$

are  $C^k$ -diffeomorphisms on  $\mathbb{R}^d$  with  $k = \bar{m}$ . Note that a  $C^k$ -diffeomorphism  $\rho$  for an integer  $k \geq 1$  means a one-to-one mapping from  $\mathbb{R}^d$  onto  $\mathbb{R}^d$  such that  $\rho$  and its inverse  $\rho^{-1}$  are continuously differentiable up to order  $k$ .

**Assumption 2.4.** The functions  $\lambda^\xi$ ,  $\lambda^\eta$  and  $\gamma$  are real-valued  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$ -measurable function on  $\Omega \times [0, T] \times \mathbb{R}^d \times Z$ . The derivatives in  $x \in \mathbb{R}^d$  of  $\lambda^\xi$ ,  $\lambda^\eta$  up to order  $\max\{\lceil m \rceil, 1\}$  exist and are continuous in  $x \in \mathbb{R}^d$  such that

$$|D^k \lambda^\xi| \leq K\bar{\xi}, \quad |D^k \lambda^\eta| \leq K\bar{\eta} \quad k = 0, 1, 2, \dots, \max\{\lceil m \rceil, 1\}.$$

The derivatives in  $x \in \mathbb{R}^d$  of  $\bar{\gamma} := \gamma - \lambda^\eta$  up to order  $\lceil m \rceil$  exist and are continuous in  $x \in \mathbb{R}^d$  such that

$$|D^l \bar{\gamma}| \leq K\bar{\eta}^2 \quad l = 0, 1, 2, \dots, \lceil m \rceil.$$

**Assumption 2.5.** The operator  $\mathcal{R}_t$  is a linear mapping from  $L_p(\mathbb{R}^d)$  into  $L_p(\mathbb{R}^d)$  for every  $t \in [0, T]$  and  $\omega \in \Omega$ , such that for every  $\varphi \in C_0^\infty$  the function  $\mathcal{R}\varphi$  is  $\mathcal{P}$ -measurable and

$$|\mathcal{R}_t \varphi|_{W_p^n} \leq K|\varphi|_{W_p^n} \quad \text{for all } \omega \in \Omega, t \in [0, T] \text{ and } n = 0, \lceil m \rceil.$$

**Assumption 2.6.** The free data  $f = (f_t)_{t \in [0, T]}$ ,  $g = (g_t^r)_{t \in [0, T]}$  and  $h = (h_t)_{t \in [0, T]}$  are  $\mathcal{P}$ -measurable processes with values in  $H_p^m$ ,  $H_p^{m+1}(l^2)$  and  $H_{p,2}^{m+1} = H_p^{m+1}(\mathcal{L}_{p,2})$ , respectively, such that almost surely  $\mathcal{K}_{p,m}^p(T) < \infty$ , where

$$\mathcal{K}_{p,m}^p(t) := \int_0^t |f_s|_{H_p^m}^p + |g_s|_{H_p^{m+1}(l^2)}^p + |h_s|_{H_{p,2}^{m+1}}^p + \mathbf{1}_{p>2} |h_s|_{H_{p,2}^{m+2}}^p ds, \quad t \leq T.$$

The initial value  $\psi$  is an  $\mathcal{F}_0$ -measurable random variable with values in  $H_p^m$ .

*Remark 2.2.* By Taylor's formula we have

$$\begin{aligned} v(x + \eta(x)) - v(x) - \eta(x) \nabla v(x) &= \int_0^1 \eta^k(x) (v_k(x + \theta\eta(x)) - v_k(x)) d\theta \\ &= \int_0^1 \eta^k(x) D_k(v(x + \theta\eta(x)) - v(x)) d\theta - \int_0^1 \theta \eta^k(x) \eta_k^l(x) v_l(x + \theta\eta(x)) d\theta, \end{aligned}$$

where to ease notation we omit the arguments  $t$  and  $z$ , and write  $v_k$  instead of  $D_k v$  for functions  $v$ . Due to Assumption 2.3 these equations extend to  $v \in W_p^1$  for  $p \geq 2$  as well. Hence after changing the order of integrals, by integration by parts we obtain

$$\int_{\mathbb{R}^d} \int_Z (v(x + \eta(x)) - v(x) - \eta(x) \nabla v(x)) \varphi(x) dz dx = -(\mathcal{J}_\eta^k v, D_k \varphi) + (\mathcal{J}_\eta^0 v, \varphi)$$

for  $\varphi \in C_0^\infty$ , with

$$\mathcal{J}_\eta^k(t) v(x) = \int_0^1 \int_Z \eta^k(v(\tau_{\theta\eta}(x)) - v(x)) \mu(dz) d\theta, \quad k = 1, 2, \dots, d, \quad (2.3)$$

$$\mathcal{J}_\eta^0(t) v(x) = - \int_0^1 \int_Z \left\{ \sum_k \eta_k^k(v(\tau_{\theta\eta}(x)) - v(x)) + \theta \eta^k(x) \eta_k^l(x) v_l(\tau_{\theta\eta}(x)) \right\} \mu(dz) d\theta, \quad (2.4)$$

where for the sake of short notation the arguments  $t, z$  of  $\eta$  and of  $\eta_k = D_k \eta$  have been omitted, and  $\tau_{\theta\eta}$  is defined in (2.2). Operators  $\mathcal{J}_\xi^k$  and  $\mathcal{J}_\xi^0$  are defined as  $\mathcal{J}_\eta^k$  and  $\mathcal{J}_\eta^0$  in (2.3) and (2.4) but with  $\xi$  everywhere in place of  $\eta$ .

**Definition 2.1.** An  $L_p$ -valued cadlag  $\mathcal{F}_t$ -adapted process  $u = (u_t)_{t \in [0, T]}$  is a *generalised solution* to equation (1.1)-(1.2) with initial value  $u_0 = \psi$ , if  $u_t \in W_p^1$  for  $P \otimes dt$ -almost every  $(\omega, t) \in \Omega \times [0, T]$ , such that almost surely  $u \in L_p([0, T], W_p^1)$  and

$$\begin{aligned} (u_t, \varphi) &= (\psi, \varphi) + \int_0^t \langle \mathcal{A}_s u_s, \varphi \rangle + (f_s, \varphi) ds + \int_0^t (\mathcal{M}_s^r u_s + g_s^r, \varphi) dw_s^r \\ &\quad + \int_0^t \int_Z (u_{s-}(\tau_{\eta_{s,z}}) - u_{s-} + \gamma_{s,z} u_{s-}(\tau_{\eta_{s,z}}) + h_s(z), \varphi) \tilde{\pi}(dz, ds) \end{aligned}$$

for all for each  $\varphi \in C_0^\infty$  and  $t \in [0, T]$ , where  $\tau_{\eta_{s,z}} = x + \eta_{s,z}(x)$ ,  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} \langle \mathcal{A}_s u_s, \varphi \rangle &:= -(a_s^{ij} D_j u_s, D_i \varphi) + (\bar{b}_s^i D_i u_s + c_s u_s, \varphi) + (\mathcal{R}_s u_s, \varphi) \\ &\quad - (\mathcal{J}_\xi^i u_s, D_i \varphi) + (\mathcal{J}_\xi^0 u_s, \varphi) - (\mathcal{J}_\eta^i u_s, D_i \varphi) + (\mathcal{J}_\eta^0 u_s, \varphi) \\ &\quad + \int_Z \int_{\mathbb{R}^d} \lambda_{s,z}^\xi(x) (u_s(x + \xi_{s,z}(x)) - u_s(x)) \varphi(x) dx \nu(dz) \\ &\quad + \int_Z \int_{\mathbb{R}^d} \lambda_{s,z}^\eta(x) (u_s(x + \eta_{s,z}(x)) - u_s(x)) \varphi(x) dx \mu(dz) \end{aligned} \quad (2.5)$$

with  $\bar{b}_s^i = b_s^i - D_j a_s^{ij}$  for all  $(s, \omega) \in [0, T] \times \Omega$ , and the stochastic integrals are Itô integrals.

The next theorem is the main result of this paper.

**Theorem 2.1.** *If Assumptions 2.1 through 2.6 hold with  $m \geq 0$ , then there is at most one generalised solution to (1.1). If Assumptions 2.1 through 2.6 hold with  $m \geq 1$ , then there is a unique generalised solution  $u = (u_t)_{t \in [0, T]}$ , which is a weakly cadlag  $H_p^m$ -valued adapted process, and it is a strongly cadlag  $H_p^s$ -valued process for any  $s \in [0, m)$ . Moreover,*

$$E \sup_{t \in [0, T]} |u_t|_{H_p^s}^q dt \leq N \left( E |\psi|_{H_p^s}^q + EK_{p,s}^q(T) \right) \quad \text{for } s \in [0, m], q \in (0, p] \quad (2.6)$$

with a constant  $N = N(d, m, p, q, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ .

### 3. PRELIMINARIES

For vectors  $v = (v^1, \dots, v^d) \in \mathbb{R}^d$  we define the operators  $T^v$ ,  $I^v$  and  $J^v$  by

$$\begin{aligned} T^v \varphi(x) &= \varphi(x + v), \quad I^v \varphi(x) = \varphi(x + v) - \varphi(x), \\ J^v \phi(x) &= \phi(x + v) - \phi(x) - v^i D_i \phi(x) \quad \text{for } x \in \mathbb{R}^d, \end{aligned} \quad (3.1)$$

acting on functions  $\varphi$  and  $\phi$  defined on  $\mathbb{R}^d$  such that the generalised derivative  $D_i \phi$  exist. If  $v = v(x)$  is a function of  $x \in \mathbb{R}^d$  then the notation  $T^v$ ,  $I^v$  and  $J^v$  mean the operators defined by (3.1) with  $v(x)$  in place  $v$ . For example,  $J^\xi$  and  $J^\eta$  mean for each  $\omega \in \Omega$ ,  $t \in [0, T]$  and  $z \in Z$  the operators defined on differentiable functions  $\varphi$  on  $\mathbb{R}^d$  by

$$\begin{aligned} J^\xi \varphi(x) &= \varphi(x + \xi(x)) - \varphi(x) - \eta^i(x) D_i \varphi(x), \\ J^\eta \varphi(x) &= \varphi(x + \eta(x)) - \varphi(x) - \eta^i(x) D_i \varphi(x), \quad x \in \mathbb{R}^d \end{aligned}$$



for each fixed variable  $(\omega, t, z)$  suppressed in this notation. We will often use the Taylor formulas

$$I^v \varphi(x) = \int_0^1 \varphi_i(x + \theta v) v^i d\theta \quad (3.2)$$

and

$$J^v \phi(x) = \int_0^1 (1 - \theta) \phi_{ij}(x + \theta v) v^i v^j d\theta \quad (3.3)$$

with  $\varphi_i := D_i \varphi$  and  $\phi_{ij} := D_i D_j \phi$ , which hold for every  $x \in \mathbb{R}^d$  when  $\varphi$  and  $\phi$  are continuous functions on  $\mathbb{R}^d$  with continuous derivatives up to first order and up to second order, respectively. These equations hold for  $dx$ -almost every  $x \in \mathbb{R}^d$  when  $\varphi \in W_p^1$  and  $\phi \in W_p^2$ .

We fix a non-negative smooth function  $k = k(x)$  with compact support on  $\mathbb{R}^d$  such that  $k(x) = 0$  for  $|x| \geq 1$ ,  $k(-x) = k(x)$  for  $x \in \mathbb{R}^d$ , and  $\int_{\mathbb{R}^d} k(x) dx = 1$ . For  $\varepsilon > 0$  and locally integrable functions  $v$  of  $x \in \mathbb{R}^d$  we use the notation  $v^{(\varepsilon)}$  for the mollification of  $v$ , defined by

$$v^{(\varepsilon)}(x) := S^\varepsilon v(x) := \varepsilon^{-d} \int_{\mathbb{R}^d} v(y) k((x - y)/\varepsilon) dy, \quad x \in \mathbb{R}^d. \quad (3.4)$$

Note that if  $v = v(x)$  is a locally Bochner integrable function on  $\mathbb{R}^d$ , with respect to the Lebesgue measure, which takes values in a separable Banach space, then the mollification of  $v$  is defined as in (3.4) in the sense of Bochner integral.

The following lemmas are taken from [9] and for their proof we refer to [9].

**Lemma 3.1.** *Let Assumption 2.3 hold. Then for every  $(\omega, t, z) \in \Omega \times [0, T] \times Z$  the operators  $T^\eta$ ,  $I^\eta$  and  $J^\eta$  are bounded linear operators from  $W_p^k$  to  $W_p^k$ , from  $W_p^{k+1}$  to  $W_p^k$  and from  $W_p^{k+2}$  to  $W_p^k$  respectively, for  $k = 0, 1, \dots, \bar{m}$ , such that  $T^\eta \varphi$ ,  $I^\eta f$  and  $J^\eta g$  are  $\mathcal{P} \otimes \mathcal{Z}$ -measurable  $W_p^k$ -valued functions of  $(\omega, t, z)$ , and*

$$|T^\eta \varphi|_{W_p^k} \leq N |\varphi|_{W_p^k}, \quad |I^\eta f|_{W_p^k} \leq N \bar{\eta}(z) |f|_{W_p^{k+1}}, \quad |J^\eta g|_{W_p^k} \leq N \bar{\eta}^2(z) |g|_{W_p^{k+2}}$$

for all  $\varphi \in W_p^k$ ,  $f \in W_p^{k+1}$  and  $g \in W_p^{k+2}$ , where  $N$  is a constant only depending on  $K, m, d, p$ .

**Lemma 3.2.** *Let  $\rho$  be a  $C^k(\mathbb{R}^d)$ -diffeomorphism for some  $k \geq 1$ , such that*

$$M \leq |\det D\rho| \text{ and } |D^l \rho| \leq N \quad \text{for } l = 1, 2, \dots, k \quad (3.5)$$

for some constants  $M > 0$  and  $N > 0$ . Then there are positive constants  $M' = M'(N, d)$  and  $N' = N'(N, M, d, k)$  such that (3.5) holds with  $g := \rho^{-1}$ , the inverse of  $\rho$ , in place of  $\rho$ , with  $M'$  and  $N'$  in place of  $M$  and  $N$ , respectively.

The following lemma is a slight generalisation of Lemma 3.4 in [9].

**Lemma 3.3.** *Let  $\rho$  be a  $C^k$ -diffeomorphism for  $k \geq 2$ , such that (3.5) holds for some positive constants  $M$  and  $N$ . Then there is a positive constant  $\varepsilon_0 = \varepsilon_0(M, N, d)$  such that  $\rho_{\varepsilon, \vartheta} := \vartheta \rho + (1 - \vartheta) \rho^{(\varepsilon)}$  is a  $C^k$ -diffeomorphism for every  $\varepsilon \in (0, \varepsilon_0)$  and  $\vartheta \in [0, 1]$ , and (3.5) remains valid for  $\rho_{\varepsilon, \vartheta}$  in place of  $\rho$ , with  $M'' = M/2$  in place of  $M$ . Moreover,  $\rho^{(\varepsilon)}$  is a  $C^\infty$ -diffeomorphism for  $\varepsilon \in (0, \varepsilon_0)$ .*

*Proof.* We show first that  $|\det D\rho_{\varepsilon,\vartheta}|$  is separated away from zero for sufficiently small  $\varepsilon > 0$ . To this end observe that for bounded Lipschitz functions  $v = (v^1, v^2, \dots, v^d)$  on  $\mathbb{R}^d$  and  $v_{\varepsilon,\vartheta} := \vartheta v + (1 - \vartheta)v^{(\varepsilon)}$  we have

$$|\Pi_{i=1}^d v^i - \Pi_{i=1}^d v_{\varepsilon,\vartheta}^i| \leq \sum_{i=1}^d K^{d-1} |v^i - v_{\varepsilon,\vartheta}^i| \leq K^{d-1} L\varepsilon \quad \text{for any } \varepsilon > 0 \text{ and } \vartheta \in [0, 1],$$

where  $L$  is the Lipschitz constant of  $v$  and  $K$  is a bound for  $|v|$ . Using this observation and taking into account that  $D_i \rho^l$  is bounded by  $N$  and it is Lipschitz continuous with a Lipschitz constant not larger than  $N$ , we get

$$|\det D\rho - \det D\rho_{\varepsilon,\vartheta}| \leq d! N^d \varepsilon.$$

Thus setting  $\varepsilon' = M/(2d! N^d)$ , for  $\varepsilon \in (0, \varepsilon')$  and  $\vartheta \in [0, 1]$  we have

$$\begin{aligned} |\det D\rho_{\varepsilon,\vartheta}| &\geq |\det D\rho| - |\det D\rho - \det D\rho_{\varepsilon,\vartheta}| \\ &\geq |\det D\rho|/2 \geq M/2. \end{aligned}$$

Clearly,  $\rho_{\varepsilon,\vartheta}$  is a  $C^k$  function. Hence by the implicit function theorem  $\rho_{\varepsilon,\vartheta}$  is a local  $C^k$ -diffeomorphism for  $\varepsilon \in (0, \varepsilon')$  and  $\vartheta \in [0, 1]$ . We prove now that  $\rho_{\varepsilon,\vartheta}$  is a global  $C^k$ -diffeomorphism for sufficiently small  $\varepsilon$ . Since by the previous lemma  $|D\rho^{-1}| \leq N'$ , we have

$$\begin{aligned} |x - y| &\leq N' |\rho(x) - \rho(y)| \\ &\leq N' |\rho_{\varepsilon,\vartheta}(x) - \rho_{\varepsilon,\vartheta}(y)| + N' |\rho(x) - \rho_{\varepsilon,\vartheta}(x) + \rho_{\varepsilon,\vartheta}(y) - \rho(y)| \end{aligned}$$

for all  $x, y \in \mathbb{R}^d$  and  $\varepsilon > 0$  and  $\vartheta \in [0, 1]$ . Observe that

$$\begin{aligned} |\rho(x) - \rho_{\varepsilon,\vartheta}(x) + \rho_{\varepsilon,\vartheta}(y) - \rho(y)| &\leq \int_{\mathbb{R}^d} |\rho(x) - \rho(x - \varepsilon u) + \rho(y - \varepsilon u) - \rho(y)| k(u) du \\ &\leq \int_{\mathbb{R}^d} \int_0^1 \varepsilon |u| |\nabla \rho(x - \theta \varepsilon u) - \nabla \rho(y - \theta \varepsilon u)| k(u) d\theta du \\ &\leq \varepsilon N |x - y| \int_{|u| \leq 1} |u| k(u) du \leq \varepsilon N |x - y|. \end{aligned}$$

Thus  $|x - y| \leq N' |\rho_{\varepsilon,\vartheta}(x) - \rho_{\varepsilon,\vartheta}(y)| + \varepsilon N' N |x - y|$ . Therefore setting  $\varepsilon'' = 1/(2N N')$ , for all  $\varepsilon \in (0, \varepsilon'')$  and  $\vartheta \in [0, 1]$  we have

$$|x - y| \leq 2N' |\rho_{\varepsilon,\vartheta}(x) - \rho_{\varepsilon,\vartheta}(y)| \quad \text{for all } x, y \in \mathbb{R}^d,$$

which implies  $\lim_{|x| \rightarrow \infty} |\rho_{\varepsilon,\vartheta}(x)| = \infty$ , i.e., under  $\rho_{\varepsilon,\vartheta}$  the pre-image of any compact set is a compact set for each  $\varepsilon \in (0, \varepsilon'')$  and  $\vartheta \in [0, 1]$ . A continuous function with this property is called a *proper function*, and by Theorem 1 in [13] a local  $C^1$ -diffeomorphism from  $\mathbb{R}^d$  into  $\mathbb{R}^d$  is a global diffeomorphism if and only if it is a proper function. Thus we have proved that  $\rho_{\varepsilon,\vartheta}$  is a global  $C^k$ -diffeomorphism for  $\varepsilon \in (0, \varepsilon_0)$  and  $\vartheta \in [0, 1]$ , where  $\varepsilon_0 = \min(\varepsilon', \varepsilon'')$ . Clearly,  $\rho_{\varepsilon,0} = \rho^{(\varepsilon)}$  is a  $C^\infty$  function and hence it is a  $C^\infty$ -diffeomorphism for every  $\varepsilon \in (0, \varepsilon_0)$ .

Now we can complete the proof of the lemma by noting that since  $D_j \rho^{(\varepsilon)} = (D_j \rho)^{(\varepsilon)}$ , the condition  $|D^i \rho| \leq N$  implies  $|D^i \rho_{\varepsilon,\vartheta}| \leq N$  for any  $\varepsilon > 0$  and  $\vartheta \in [0, 1]$ .  $\square$

For fixed  $\varepsilon > 0$  and  $\vartheta \in [0, 1]$  let  $\rho_{\varepsilon, \vartheta}$  denote any of the functions

$$\rho_{\varepsilon, \vartheta}(x) := x + \vartheta \eta_{t,z}(x) + (1 - \vartheta) \eta_{t,z}^{(\varepsilon)}(x), \quad \rho_{\varepsilon, \vartheta}(x) := x + \vartheta \xi_{t,z}(x) + (1 - \vartheta) \xi_{t,z}^{(\varepsilon)}(x) \quad x \in \mathbb{R}^d.$$

for each  $(\omega, t, z) \in \Omega \times [0, T] \times Z$ , and assume that Assumptions 2.2 and 2.3 hold. Then by the inverse function theorem  $\rho$  is a local  $C^1$ -diffeomorphism for each  $t, \theta$  and  $z$ . Since

$$|\eta_{t,z}(x)| \leq \bar{\eta}(z) < \infty, \quad |\xi_{t,z}(x)| \leq \bar{\xi}(z) < \infty,$$

we have  $\lim_{|x| \rightarrow \infty} |\rho_{\varepsilon, \vartheta}(x)| = \infty$ . Hence  $\rho_{\varepsilon, \vartheta}$  is a global  $C^1$ -diffeomorphism on  $\mathbb{R}^d$ , for  $\varepsilon > 0$ ,  $\vartheta \in [0, 1]$ , for each  $t \in [0, T]$ ,  $z \in Z$  and  $\theta \in [0, 1]$ , by Theorem 1 in [13]. Note that by the formula on the derivative of inverse functions, a  $C^1$ -diffeomorphism and its inverse have continuous derivatives up to the same order. Thus Lemmas 3.2 and 3.3 imply the following statement, which is a slight generalisation of Corollary 3.6 in [9].

**Corollary 3.4.** *Let Assumptions 2.2 and 2.3 hold. Then there is a positive constant  $\varepsilon_0 = \varepsilon_0(K, d)$  such that  $\rho = \rho_{\varepsilon, \vartheta}$  is a  $C^k$ -diffeomorphism on  $\mathbb{R}^d$  for  $k = \bar{m}$ , for any  $\varepsilon \in (0, \varepsilon_0)$ ,  $\vartheta \in [0, 1]$  and  $(\omega, t, z) \in \Omega \times [0, T] \times Z$ . Moreover, for some constants  $M = M(K, d, \bar{m})$  and  $N = N(K, d, \bar{m})$*

$$M \leq \min(|\det D\rho|, |\det(D\rho)^{-1}|), \quad \max(|D^k \rho|, |D^k \rho^{-1}|) \leq N \quad (3.6)$$

for any  $\varepsilon \in (0, \varepsilon_0)$ ,  $\vartheta \in [0, 1]$ ,  $(\omega, t, z) \in \Omega \times [0, T] \times Z$  and for  $k = 1, 2, \dots, \bar{m}$ . Furthermore, if  $\vartheta = 0$  then  $\rho$  is a  $C^\infty$ -diffeomorphism for each  $\varepsilon \in (0, \varepsilon_0)$ ,  $(\omega, t, z) \in \Omega \times [0, T] \times Z$ , and for each integer  $m \geq 1$  there are constants  $M = M(K, d, m)$  and  $N = N(K, d, m)$  such the estimates in (3.6) hold for all  $\varepsilon \in (0, \varepsilon_0)$ ,  $(\omega, t, z) \in \Omega \times [0, T] \times Z$  and  $k = 1, 2, \dots, m$ .

**Lemma 3.5.** *Let  $V$  be a separable Banach space, and let  $f = f(x)$  be a  $V$ -valued function of  $x \in \mathbb{R}^d$  such that  $f \in L_p(V) = L_p(\mathbb{R}^d, V)$  for some  $p \geq 1$ . Then we have*

$$|f^{(\varepsilon)}|_{L_p(V)} \leq |f|_{L_p(V)} \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} |f^{(\varepsilon)} - f|_{L_p(V)} = 0.$$

*Proof.* This lemma is well-known. Its proof can be found, e.g., in [17], see Lemma 4.4 therein. For the convenience of the readers, we present the proof below. By the properties of Bochner integrals, Jensen's inequality and Fubini's theorem

$$\begin{aligned} |f^{(\varepsilon)}|_{L_p(V)}^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) k_\varepsilon(x - y) dy \right|_V^p dx \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)|_V^p k_\varepsilon(x - y) dy dx = |f|_{L_p(V)}^p. \end{aligned}$$

Since  $V$  is separable, it has a countable dense subset  $V_0$ . Denote by  $\mathcal{H} \subset L_p(V)$  the space of functions  $h$  of the form

$$h(x) = \sum_{i=1}^k v_i \varphi_i(x)$$

for some integer  $k \geq 1$ ,  $v_i \in V_0$  and continuous real functions  $\varphi_i$  on  $\mathbb{R}^d$  with compact support. Then for such an  $h$  we have

$$|h^{(\varepsilon)} - h|_{L_p(V)} \leq \sum_{i=1}^k |\varphi_i^{(\varepsilon)} - \varphi_i|_{L_p} |v_i|_V \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where  $L_p = L_p(\mathbb{R}^d, \mathbb{R})$ . For  $f \in L_p(V)$  and  $h \in \mathcal{H}$  we have

$$|f - f^{(\varepsilon)}|_{L_p(V)} \leq |f - h|_{L_p(V)} + |h - h^{(\varepsilon)}|_{L_p(V)} + |(f - h)^{(\varepsilon)}|_{L_p(V)} \leq 2|f - h|_{L_p(V)} + |h - h^{(\varepsilon)}|_{L_p(V)}.$$

Letting here  $\varepsilon \rightarrow 0$  for each  $f \in L_p(V)$  we obtain

$$\limsup_{\varepsilon \rightarrow 0} |f - f^{(\varepsilon)}|_{L_p(V)} \leq 2|f - h|_{L_p(V)} \quad \text{for all } h \in \mathcal{H}.$$

Since  $\mathcal{H}$  is dense in  $L_p(V)$ , we can choose  $h \in \mathcal{H}$  to make  $|f - h|_{L_p(V)}$  arbitrarily small, which proves  $\lim_{\varepsilon \rightarrow 0} |f - f^{(\varepsilon)}|_{L_p(V)} = 0$ .  $\square$

Recall that  $L_p(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$  denotes the  $L_p$ -space of  $\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2}$ -valued functions on  $\mathbb{R}^d$  with respect to the Lebesgue measure on  $\mathbb{R}^d$ . Since  $(Z, \mathcal{Z}, \mu)$  is a  $\sigma$ -finite separable measure space,  $\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2}$  is a separable Banach space for any  $q_1, q_2 \in [1, \infty)$ . Hence, by Lemma 3.6 in [17] for each  $v \in L_p(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$ ,  $p > 1$  there is a  $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$ -measurable function  $\bar{v} = \bar{v}(x, z)$  such that for every  $x \in \mathbb{R}^d$  we have  $v(x, z) = \bar{v}(x, z)$  for  $\mu$ -almost every  $z \in Z$ . Therefore if  $v \in L_p(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$  for some  $p > 1$ , then we may assume that  $v$  is a  $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$ -measurable real-valued function. Moreover, we will often use the following characterisation of  $W_p^n(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$ .

**Lemma 3.6.** *Let  $v \in L_p(\mathcal{L}_p \cap \mathcal{L}_q)$  for some  $p, q \in (1, \infty)$ , and let  $\alpha$  be a multi-index. Then the following statements hold.*

- (i) *If  $v_\alpha$ , the  $\mathcal{L}_p \cap \mathcal{L}_q$ -valued generalised  $D_\alpha$ -derivative, belongs to  $L_p(\mathcal{L}_p \cap \mathcal{L}_q)$ , then for  $\mu$ -almost every  $z \in Z$  the function  $v_\alpha(\cdot, z)$  belongs to  $L_p(\mathbb{R}^d, \mathbb{R})$  and it is the generalised  $D_\alpha$ -derivative of  $v(\cdot, z)$ .*
- (ii) *If  $v_\alpha(\cdot, z)$ , the generalised  $D_\alpha$ -derivative of the function  $v(\cdot, z)$ , belongs to  $L_p(\mathbb{R}^d, \mathbb{R})$  for  $\mu$ -almost every  $z \in Z$  such that*

$$\int_{\mathbb{R}^d} \left( \int_Z |v_\alpha(x, z)|^r \mu(dz) \right)^{p/r} dx < \infty \quad \text{for } r = p, q, \quad (3.7)$$

*then  $v_\alpha$  belongs to  $L_p(\mathcal{L}_p \cap \mathcal{L}_q)$ , and it is the  $\mathcal{L}_p \cap \mathcal{L}_q$ -valued generalised  $D_\alpha$ -derivative of  $v$ .*

*Proof.* (i) Let  $\bar{v}_\alpha$  denote the  $\mathcal{L}_p \cap \mathcal{L}_q$ -valued generalised  $D_\alpha$ -derivative of  $v$ . Then

$$\int_{\mathbb{R}^d} \bar{v}_\alpha(x) \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x) D_\alpha \varphi(x) dx$$

for every  $\varphi \in C_0^\infty$ , where the integrals are understood as Bochner integrals of  $\mathcal{L}_p \cap \mathcal{L}_q$ -valued functions. Hence

$$\int_{\mathbb{R}^d} \int_Z \bar{v}_\alpha(x, z) \psi(z) \varphi(x) \mu(dz) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \int_Z \bar{v}(x, z) \psi(z) \varphi_\alpha(x) \mu(dz) dx$$

for all  $\varphi \in C_0^\infty$  and bounded  $\mathcal{Z}$ -measurable functions  $\psi$  supported on sets of finite  $\mu$ -measure. We can use Fubini's theorem to get

$$\int_Z \int_{\mathbb{R}^d} \bar{v}_\alpha(x, z) \varphi(x) dx \psi(z) \mu(dz) = (-1)^{|\alpha|} \int_Z \int_{\mathbb{R}^d} v(x, z) \varphi_\alpha(x) dx \psi(z) \mu(dz).$$

Thus for each  $\varphi \in C_0^\infty$  we have

$$\int_{\mathbb{R}^d} \bar{v}_\alpha(x, z) \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x, z) \varphi_\alpha(x) dx \quad (3.8)$$

for  $\mu$ -almost every  $z \in Z$ . Consequently, for  $\mu$ -almost every  $z \in Z$  equation (3.8) holds for all  $\varphi \in \Phi$  for a separable dense set  $\Phi \subset C_0^\infty$  in  $L_{p/(p-1)}(\mathbb{R}^d, \mathbb{R})$ . Notice that for  $\mu$ -almost every  $z \in Z$  the functions  $\bar{v}_\alpha(\cdot, z)$  and  $v(\cdot, z)$  belong to  $L_p(\mathbb{R}^d, \mathbb{R})$ . Hence there is a set  $S \subset Z$  of full  $\mu$ -measure such that for  $z \in S$  equation (3.8) holds for all  $\varphi \in C_0^\infty$ , which proves that for  $z \in S$  the function  $\bar{v}_\alpha(\cdot, z)$  is the generalised  $D_\alpha$ -derivative of  $v(\cdot, z)$ . To prove (ii) notice that if for  $\mu$ -almost every  $z \in Z$  the function  $v_\alpha(\cdot, z)$  belongs to  $L_p(\mathbb{R}^d, \mathbb{R})$  and it is the  $D_\alpha$  generalised derivative of the function  $v(\cdot, z)$ , then for  $\mu$ -almost every  $z \in Z$  we have

$$\int_{\mathbb{R}^d} v_\alpha(x, z) \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x, z) \varphi(x) dx$$

for every  $\varphi \in C_0^\infty$ . Using condition (3.7) and that  $v \in L_p(\mathcal{L}_p \cap \mathcal{L}_q)$ , it is easy to check that, as functions of  $z$ , both sides of the above equation are functions in  $\mathcal{L}_p \cap \mathcal{L}_q$ , and hence that these integrals define the same functions as the corresponding  $\mathcal{L}_p \cap \mathcal{L}_q$ -valued Bochner integrals. This proves that  $v_\alpha$  is the  $\mathcal{L}_p \cap \mathcal{L}_q$ -valued generalised  $D_\alpha$ -derivative of  $v$ .  $\square$

**Lemma 3.7.** *Let Assumptions 2.3 hold with  $m = 0$ . Then the following statements hold.*

- (i) *Let  $\zeta$  be a  $\mathcal{F} \otimes \mathcal{B}([0, T] \times \mathbb{R}^d) \otimes \mathcal{Z}$ -measurable function on  $\Omega \times H_T \times Z$  such that it is continuously differentiable in  $x \in \mathbb{R}^d$  and*

$$|\zeta| + |D\zeta| \leq K\bar{\eta} \quad \text{for all } (\omega, t, x, z) \in \Omega \times H_T \times Z. \quad (3.9)$$

*Then there is a constant  $N = N(K, d)$  such that for  $\varphi \in W_1^1$*

$$A := \int_{\mathbb{R}^d} \zeta(t, x, z) I^\eta \varphi(x) dx \leq N\bar{\eta}^2(z) |\varphi|_{L_1} \quad \text{for all } (\omega, t, z) \in \Omega \times [0, T] \times Z. \quad (3.10)$$

- (ii) *There is a constant  $N = N(K, d)$  such that for all  $\phi \in W_1^2$*

$$B := \int_{\mathbb{R}^d} J^\eta \phi(x) dx \leq N\bar{\eta}^2(z) |\phi|_{L_1}. \quad (3.11)$$

- (iii) *There is a constant  $N = N(K, d)$  such that for all  $\phi \in W_1^1$*

$$C := \int_{\mathbb{R}^d} I^\eta \phi(x) dx \leq N\bar{\eta}(z) |\phi|_{L_1}. \quad (3.12)$$

*Proof.* The proof of (3.11) and (3.12) is given in [8] and [9]. For the convenience of the reader we prove each of the above estimates here. We may assume that  $\varphi, \phi \in C_0^\infty$ . For each  $(\omega, t, z, \theta) \in \Omega \times [0, T] \times Z \times [0, 1]$  let  $\tau_{\theta\eta}^{-1}$  denote the inverse of the function  $x \rightarrow x + \theta\eta_{t,z}(x)$ . Using (3.2) and (3.3) by change of variables we have

$$A = \int_0^1 \int_{\mathbb{R}^d} \nabla \varphi(x) \chi_{t,z,\theta}(x) dx d\theta, \quad B = \int_0^1 \int_{\mathbb{R}^d} (1 - \theta) D_{ij} \phi(x) \varrho_{t,z,\theta}^{ij}(x) dx d\theta \quad (3.13)$$

$$C = \int_0^1 \int_{\mathbb{R}^d} \nabla \phi(x) \kappa_{t,z,\theta}(x) dx d\theta \quad (3.14)$$

with

$$\chi(x) := (\zeta\eta)(\tau_{\theta\eta}^{-1}(x)) |\det D\tau_{\theta\eta}^{-1}(x)|, \quad \varrho^{ij}(x) := (\eta^i \eta^j)(\tau_{\theta\eta}^{-1}(x)) |\det D\tau_{\theta\eta}^{-1}(x)|$$

and

$$\kappa(x) := \eta(\tau_{\theta\eta}^{-1}(x)) |\det D\tau_{\theta\eta}^{-1}(x)|.$$

Due to (3.9) and Assumption 2.3 we have a constant  $N = N(K, d)$  such that

$$|D\chi_{t,\theta,z}(x)| \leq N\bar{\eta}^2(z), \quad |D_{ij}\partial_{t,z,\theta}^{ij}(x)| \leq N\bar{\eta}^2(z) \quad \text{and} \quad |D\kappa_{t,\theta,z}(x)| \leq N\bar{\eta}(z)$$

for all  $(\omega, x, t, z, \theta) \in \Omega \times \mathbb{R}^d \times [0, T] \times Z \times [0, 1]$ . Thus from (3.13) and (3.14) by integration by parts we get (3.10), (3.11) and (3.12).  $\square$

**Lemma 3.8.** *Let Assumption 2.3 hold with an integer  $m \geq 0$ . Then for all  $\omega \in \Omega$ ,  $t \in [0, T]$  and  $r > 1$  we have*

$$\mathbf{H}(\varphi, h) := \int_Z \left| \int_{\mathbb{R}^d} T^\eta \varphi(x) h(x, z) dx \right| \mu(dz) \leq N |\varphi|_{L_{r/(r-1)}} |h|_{W_r^2(\mathcal{L}_{r,2})}, \quad (3.15)$$

$$\mathbf{K}(\phi, g) := \int_Z \left| \int_{\mathbb{R}^d} T^\eta \phi(x) g(x, z) dx \right| \mu(dz) \leq N |\phi|_{L_{r/(r-1)}} |g|_{W_r^2(\mathcal{L}_{r,1})} \quad (3.16)$$

for  $\varphi, \phi \in L_{r/(r-1)}$  and  $h \in W_r^2(\mathcal{L}_{r,2})$  and  $g \in W_r^2(\mathcal{L}_{r,1})$  with a constant  $N = N(r, d, K, K_{\bar{\eta}})$ .

*Proof.* First assume  $\varphi, \phi \in C_0^\infty$ , and notice that

$$\mathbf{H}(\varphi, h) \leq \sum_{j=0}^1 \mathbf{H}^{(j)} \quad \text{and} \quad \mathbf{K}(\phi, h) \leq \sum_{j=0}^2 \mathbf{K}^{(j)}$$

with

$$\mathbf{H}^{(0)} = \int_Z \left| \int_{\mathbb{R}^d} J^\eta \varphi h dx \right| \mu(dz), \quad \mathbf{H}^{(1)} = \int_Z \left| \int_{\mathbb{R}^d} D_i \varphi \eta^i h dx \right| \mu(dz)$$

and

$$\mathbf{K}^{(0)} = \int_Z \left| \int_{\mathbb{R}^d} J^\eta \phi g dx \right| \mu(dz), \quad \mathbf{K}^{(1)} = \int_Z \left| \int_{\mathbb{R}^d} D_i \phi \eta^i g dx \right| \mu(dz),$$

$$\mathbf{K}^{(2)} = \int_Z \left| \int_{\mathbb{R}^d} \phi g dx \right| \mu(dz).$$

By Fubini's theorem and Hölder's inequality

$$\mathbf{K}^{(2)} \leq \int_{\mathbb{R}^d} |\phi| |g(x, \cdot)|_{\mathcal{L}_1} dx \leq |\phi|_{L_{r/(r-1)}} |g|_{L_r(\mathcal{L}_1)}. \quad (3.17)$$

By Taylor's formula, Fubini's theorem, change of variables, integration by parts and using Assumption 2.3 we get

$$\begin{aligned} \mathbf{H}^{(0)} &\leq \int_0^1 \int_Z \left| \int_{\mathbb{R}^d} (1-\theta) (T^{\theta\eta} \varphi_{ij}) \eta^i \eta^j h dx \right| \mu(dz) d\theta \\ &\leq N \int_0^1 \int_Z \int_{\mathbb{R}^d} \bar{\eta}^2(z) |\varphi| \sum_{|\beta| \leq 2} |h_\beta| (\tau_{\theta\eta}^{-1}(x), z) dx \mu(dz) d\theta \end{aligned}$$

with  $N = N(d, K)$ , where  $\tau_{\theta\eta}$  is defined by (2.2), and  $\tau_{\theta\eta}^{-1}$  is its inverse. Hence by Hölder's inequality, change of variables, Fubini's theorem and using  $|\bar{\eta}| \leq K$  we obtain

$$\mathbf{H}^{(0)} \leq N |\varphi|_{L_{r/(r-1)}} |h|_{W_r^2(\mathcal{L}_r)}, \quad (3.18)$$

and in the same way

$$\mathbf{K}^{(0)} \leq N |\phi|_{L_{r/(r-1)}} |g|_{W_r^2(\mathcal{L}_r)} \quad (3.19)$$

with a constant  $N = N(K, d, r, K_{\bar{\eta}})$ . By integration by parts, using Assumption 2.3, Cauchy-Schwarz and Hölder inequalities we get

$$\begin{aligned} \mathbf{H}^{(1)} &= \int_Z \left| \int_{\mathbb{R}^d} \varphi D_i(\eta^i h(x, z)) dx \right| \mu(dz) \leq \int_Z \int_{\mathbb{R}^d} |\varphi| \bar{\eta}(z) \sum_{|\beta| \leq 1} |h_\beta(x, z)| dx \mu(dz) \\ &\leq K_{\bar{\eta}} \int_{\mathbb{R}^d} |\varphi| \sum_{|\beta| \leq 1} |h_\beta(x, \cdot)|_{\mathcal{L}_2} dx \leq N |\varphi|_{L_{r/(r-1)}} |h|_{W_r^2(\mathcal{L}_{r,2})} \end{aligned} \quad (3.20)$$

and in the same way

$$\mathbf{K}^{(1)} \leq N |\phi|_{L_{r/(r-1)}} |g|_{W_r^1(\mathcal{L}_1)}$$

with a constant  $N = N(K, d, r, K_{\bar{\eta}})$ . Combining this with (3.17) through (3.20), we get (3.15) and (3.16) for  $\varphi, \phi \in C_0^\infty$ . Assume now that  $\varphi, \phi \in L_{r/(r-1)}$  and  $\mu(Z) < \infty$ . It is easy to see that by Hölder inequality we have

$$\mathbf{H}(\varphi, h) \leq C |\varphi|_{L_{r/(r-1)}} |h|_{L_r(\mathcal{L}_r)} \quad \text{and} \quad \mathbf{K}(\phi, g) \leq C |\phi|_{L_{r/(r-1)}} |g|_{L_r(\mathcal{L}_r)}$$

for  $\varphi, \phi \in L_{r/(r-1)}$  and  $h, g \in L_r(\mathcal{L}_r)$  with a constant  $C = C(K, K_{\bar{\eta}}, d, r, \mu(Z))$ , which implies that estimates (3.15) and (3.16) for  $\varphi, \phi \in C_0^\infty$  can be extended by continuity to  $\varphi, \phi \in L_{r/(r-1)}$  for finite measures  $\mu$ . In the general case of a  $\sigma$ -finite measure  $\mu$ , there exist  $Z_n \in \mathcal{Z}$ ,  $n = 1, 2, \dots$ , such that  $\mu(Z_n) < \infty$  and  $\cup_{n=1}^\infty Z_n = Z$ . We define measures  $\mu_n$  for integers  $n \geq 1$  such that  $d\mu_n = \mathbf{1}_{Z_n} d\mu$ , where  $\mathbf{1}_{Z_n}$  is the indicator for  $Z_n$ . By the previous argument, for  $\varphi, \phi \in L_{r/(r-1)}$  we have

$$\int_Z \left| \int_{\mathbb{R}^d} I^\eta \varphi(x) h(x, z) dx \right| \mu_n(dz) \leq N |\varphi|_{L_{r/(r-1)}} |h|_{W_r^2(\mathcal{L}_{r,2})}^r$$

and

$$\int_Z \left| \int_{\mathbb{R}^d} T^\eta \phi(x) g(x, z) dx \right| \mu_n(dz) \leq N |\phi|_{L_{r/(r-1)}} |g|_{W_r^2(\mathcal{L}_{r,1})}$$

with a constant  $N = N(r, d, K, K_{\bar{\eta}})$ . Then an application of Fatou's lemma finishes the proof of this lemma.  $\square$

Next we present two important Itô's formulas from [17] for the  $p$ -th power of  $L_p$ -norm of  $L_p$ -valued stochastic processes.

**Lemma 3.9.** *Let  $(u_t^i)_{t \in [0, T]}$  be a progressively measurable  $L_p$ -valued process such that there exist  $f^i \in \mathbb{L}_p$ ,  $g^i = (g^{ir})_{r=1}^\infty \in \mathbb{L}_p$ ,  $h^i \in \mathbb{L}_{p,2}$ , and an  $L_p$ -valued  $\mathcal{F}_0$ -measurable random variable  $\psi^i$  for each  $i = 1, 2, \dots, M$  for some integer  $M$ , such that for every  $\varphi \in C_0^\infty$*

$$(u_t^i, \varphi) = (\psi, \varphi) + \int_0^t (f_s^i, \varphi) ds + \int_0^t (g_s^{ir}, \varphi) dw_s^r + \int_0^t \int_Z (h_s^i, \varphi) \tilde{\pi}(dz, ds) \quad (3.21)$$

for  $P \otimes dt$ -almost every  $(\omega, t) \in \Omega \times [0, T]$  and all  $i = 1, 2, \dots, M$ . Then there are  $L_p$ -valued adapted cadlag processes  $\bar{u} = (\bar{u}^1, \bar{u}^2, \dots, \bar{u}^M)$  such that equation (3.21), with  $\bar{u}^i$  in place of  $u^i$ , holds for every  $i = 1, 2, \dots, M$  and each  $\varphi \in C_0^\infty$  almost surely for all  $t \in [0, T]$ . Moreover,  $u^i = \bar{u}^i$  for  $P \otimes dt$ -almost every  $(\omega, t) \in \Omega \times [0, T]$ , and

$$|\bar{u}_t|_{L_p}^p = |\psi|_{L_p}^p + p \int_0^t \int_{\mathbb{R}^d} |\bar{u}_s|^{p-2} \bar{u}_s^i g_s^{ir} dx dw_s^r$$

$$\begin{aligned}
 & + \frac{p}{2} \int_0^t \int_{\mathbb{R}^d} (2|\bar{u}_s|^{p-2} \bar{u}_s^i f_s^i + (p-2)|\bar{u}_s|^{p-4} |\bar{u}_s^i g_s^i|_{l_2}^2 + |\bar{u}_s|^{p-2} \sum_{i=1}^M |g_s^i|_{l_2}^2) dx ds \\
 & + p \int_0^t \int_Z \int_{\mathbb{R}^d} |\bar{u}_{s-}|^{p-2} \bar{u}_{s-}^i h_s^i dx \tilde{\pi}(dz, ds) \\
 & + \int_0^t \int_Z \int_{\mathbb{R}^d} (|\bar{u}_{s-} + h_s|^p - |\bar{u}_{s-}|^p - p|\bar{u}_{s-}|^{p-2} \bar{u}_{s-}^i h_s^i) dx \pi(dz, ds)
 \end{aligned}$$

holds almost surely for all  $t \in [0, T]$ , where the definition  $0/0 := 0$  is used where  $0/0$  occurs.

**Lemma 3.10.** *Let  $u = (u_t)_{t \in [0, T]}$  be a progressively measurable  $W_p^1$ -valued process such that the following conditions hold:*

(i)

$$E \int_0^T |u_t|_{W_p^1}^p dt < \infty;$$

(ii) *there exist  $f \in \mathbb{L}_p$  for  $\alpha \in \{0, 1, \dots, d\}$ ,  $g \in \mathbb{L}_p$ ,  $h \in \mathbb{L}_{p,2}$ , and an  $L_p$ -valued  $\mathcal{F}_0$ -measurable random variable  $\psi$ , such that for every  $\varphi \in C_0^\infty$  we have*

$$(u_t, \varphi) = (\psi, \varphi) + \int_0^t (f_s^\alpha, D_\alpha^* \varphi) ds + \int_0^t (g_s^r, \varphi) dw_s^r + \int_0^t \int_Z (h_s(z), \varphi) \tilde{\pi}(dz, ds) \quad (3.22)$$

for  $P \otimes dt$ -almost every  $(\omega, t) \in \Omega \times [0, T]$ , where  $D_\alpha^* = -D_\alpha$  for  $\alpha = 1, 2, \dots, d$ , and  $D_\alpha^*$  is the identity operator for  $\alpha = 0$ . Then there is an  $L_p$ -valued adapted cadlag process  $\bar{u} = (\bar{u}_t)_{t \in [0, T]}$  such that for each  $\varphi \in C_0^\infty$  equation (3.22) holds with  $\bar{u}$  in place of  $u$  almost surely for all  $t \in [0, T]$ . Moreover,  $u = \bar{u}$  for  $P \otimes dt$ -almost every  $(\omega, t) \in \Omega \times [0, T]$ , and almost surely

$$\begin{aligned}
 |\bar{u}_t|_{L_p}^p &= |\psi|_{L_p}^p + p \int_0^t \int_{\mathbb{R}^d} |u_s|^{p-2} u_s g_s^r dx dw_s^r \\
 &+ \frac{p}{2} \int_0^t \int_{\mathbb{R}^d} (2|u_s|^{p-2} u_s f_s^0 - 2(p-1)|u_s|^{p-2} f_s^i D_i u_s + (p-1)|u_s|^{p-2} |g_s|_{l_2}^2) dx ds \\
 &+ \int_0^t \int_Z \int_{\mathbb{R}^d} p|\bar{u}_{s-}|^{p-2} \bar{u}_{s-} h_s dx \tilde{\pi}(dz, ds) \\
 &+ \int_0^t \int_Z \int_{\mathbb{R}^d} (|\bar{u}_{s-} + h_s|^p - |\bar{u}_{s-}|^p - p|\bar{u}_{s-}|^{p-2} \bar{u}_{s-} h_s) dx \pi(dz, ds)
 \end{aligned}$$

for all  $t \in [0, T]$ , where  $\bar{u}_{s-}$  denotes the left-hand limit in  $L_p(\mathbb{R}^d)$  of  $\bar{u}$  at  $s \in (0, T]$ .

The following slight generalisation of Lemma from [16] will play a role in obtaining supremum estimates.

**Lemma 3.11.** *Let  $T \in [0, \infty]$  and let  $f = (f_t)_{t \geq 0}$  and  $g = (g_t)_{t \geq 0}$  be nonnegative  $\mathcal{F}_t$ -adapted processes such that  $f$  is a cadlag and  $g$  is a continuous process. Assume*

$$Ef_\tau \mathbf{1}_{g_0 \leq c} \leq Eg_\tau \mathbf{1}_{g_0 \leq c} \quad (3.23)$$

for any constant  $c > 0$  and bounded stopping time  $\tau \leq T$ . Then, for any bounded stopping time  $\tau \leq T$ , for  $q \in (0, 1)$

$$E \sup_{t \leq \tau} f_t^q \leq \frac{2-q}{1-q} E \sup_{t \leq \tau} g_t^q.$$



*Proof.* This lemma is proved in [16] when both processes  $f$  and  $g$  are continuous. A word by word repetition of the proof in [16] extends it to the case when  $f$  is cadlag. For the convenience of the reader we present the proof below. By replacing  $f_t$  and  $g_t$  with  $f_{t \wedge T}$  and  $g_{t \wedge T}$ , respectively, we see that we may assume that  $T = \infty$ . Then we replace  $g_t$  with  $\max_{s \leq t} g_s$  and see that without losing generality we may assume that  $g_t$  is nondecreasing. In that case fix a constant  $c > 0$  and let  $\theta_f = \inf\{t \geq 0 : f_t \geq c\}$ ,  $\theta_g = \inf\{t \geq 0 : g_t \geq c\}$ . Then

$$P(\sup_{t \leq \tau} f_t > c) \leq P(\theta_f \leq \tau) \leq P(\theta_g \leq \tau) + P(\theta_f \leq \tau \wedge \theta_g, \theta_g > \tau)$$

$$\leq P(g_\tau \geq c) + P(g_0 \leq c, f_{\tau \wedge \theta_g \wedge \theta_f} \geq c) \leq P(g_\tau \geq c) + \frac{1}{c} EI_{g_0 \leq c} f_{\tau \wedge \theta_g \wedge \theta_f}.$$

In the light of (3.23) we replace the expectation with

$$\begin{aligned} EI_{g_0 \leq c} g_{\tau \wedge \theta_g \wedge \theta_f} &\leq EI_{g_0 \leq c} g_{\tau \wedge \theta_g} = EI_{g_0 \leq c} (g_\tau \wedge g_{\theta_g}) \\ &\leq EI_{g_0 \leq c} (g_\tau \wedge c) \leq E(g_\tau \wedge c). \end{aligned}$$

Hence

$$P(\sup_{t \leq \tau} f_t > c) \leq P(g_\tau \geq c) + \frac{1}{c} E(c \wedge g_\tau),$$

and it remains to substitute  $c^{1/q}$  in place of  $c$  and integrate with respect to  $c$  over  $(0, \infty)$ . The lemma is proved.  $\square$

Finally we present a slight modification of Lemma 5.3 from [9] which we will use in proving regularity in time of solutions to (1.1)-(1.2).

**Lemma 3.12.** *Let  $V$  be a reflexive Banach space, embedded continuously and densely into a Banach space  $U$ . Let  $f$  be a  $U$ -valued weakly cadlag function on  $[0, T]$  such that the weak limit in  $U$  at  $T$  from the left is  $f(T)$ . Assume there is a dense subset  $S$  of  $[0, T]$  such that  $f(s) \in V$  for  $s \in S$  and  $\sup_{s \in S} |f(s)|_V < \infty$ . Then  $f$  is a  $V$ -valued function, which is cadlag in the weak topology of  $V$ , and hence  $\sup_{s \in [0, T]} |f(s)|_V = \sup_{s \in S} |f(s)|_V$ .*

*Proof.* Since  $S$  is dense in  $[0, T]$ , for a given  $t \in [0, T)$  there is a sequence  $\{t_n\}_{n=1}^\infty$  with elements in  $S$  such that  $t_n \downarrow t$ . Due to  $\sup_{n \in \mathbb{N}} |f(t_n)|_V < \infty$  and the reflexivity of  $V$  there is a subsequence  $\{t_{n_k}\}$  such that  $f(t_{n_k})$  converges weakly in  $V$  to some element  $v \in V$ . Since  $f$  is weakly cadlag in  $U$ , for every continuous linear functional  $\varphi$  over  $U$  we have  $\lim_{k \rightarrow \infty} \varphi(f(t_{n_k})) = \varphi(f(t))$ . Since the restriction of  $\varphi$  in  $V$  is a continuous functional over  $V$  we have  $\lim_{k \rightarrow \infty} \varphi(f(t_{n_k})) = \varphi(v)$ . Hence  $f(t) = v$ , which proves that  $f$  is a  $V$ -valued function over  $[0, T)$ . Moreover, by taking into account that

$$|f(t)|_V = |v|_V \leq \liminf_{k \rightarrow \infty} |f(t_{n_k})|_V \leq \sup_{t \in S} |f(t)|_V < \infty,$$

we obtain  $K := \sup_{t \in [0, T)} |f(s)|_V < \infty$ . Let  $\phi$  be a continuous linear functional over  $V$ . Due to the reflexivity of  $V$ , the dual  $U^*$  of the space  $U$  is densely embedded into  $V^*$ , the dual of  $V$ . Thus for  $\phi \in V^*$  and  $\varepsilon > 0$  there is  $\phi_\varepsilon \in U^*$  such that  $|\phi - \phi_\varepsilon|_{V^*} \leq \varepsilon$ . Hence for arbitrary sequence  $t_n \downarrow t$ ,  $t_n \in [0, T]$  we have

$$\begin{aligned} |\phi(f(t)) - \phi(f(t_n))| &\leq |\phi_\varepsilon(f(t) - f(t_n))| + |(\phi - \phi_\varepsilon)(f(t) - f(t_n))| \\ &\leq |\phi_\varepsilon(f(t) - f(t_n))| + \varepsilon |f(t) - f(t_n)|_V \leq |\phi_\varepsilon(f(t) - f(t_n))| + 2\varepsilon K. \end{aligned}$$

Letting here  $n \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$ , we get

$$\limsup_{n \rightarrow \infty} |\phi(f(t)) - \phi(f(t_n))| \leq 0,$$

which proves that  $f$  is right-continuous in the weak topology in  $V$ . We can prove in the same way that at each  $t \in [0, T]$  the function  $f$  has weak limit in  $V$  from the left at each  $t \in (0, T]$ , which finishes the proof of the lemma.  $\square$

#### 4. SOME RESULTS ON INTERPOLATION SPACES

A pair of complex Banach spaces  $A_0$  and  $A_1$ , which are continuously embedded into a Hausdorff topological vector space  $\mathcal{H}$ , is called an interpolation couple, and  $A_\theta = [A_0, A_1]_\theta$  denotes the complex interpolation space between  $A_0$  and  $A_1$  with parameter  $\theta \in (0, 1)$ . For an interpolation couple  $A_0$  and  $A_1$  the notations  $A_0 \cap A_1$  and  $A_0 + A_1$  is used for the subspaces

$$A_0 \cap A_1 = \{v \in \mathcal{H} : v \in A_0 \text{ and } v \in A_1\}, \quad A_0 + A_1 = \{v \in \mathcal{H} : v = v_1 + v_2, v_i \in A_i\}$$

equipped with the norms  $|v|_{A_0 \cap A_1} = \max(|v|_{A_0}, |v|_{A_1})$  and

$$|v|_{A_0 + A_1} := \inf\{|v_0|_{A_0} + |v_1|_{A_1} : v = v_0 + v_1, v_0 \in A_0, v_1 \in A_1\},$$

respectively. Then the following theorem lists some well-known facts about complex interpolation, see e.g., [5] or 1.9.3, 1.18.4 and 2.4.2 in [34] and 5.6.9 in [18].

**Theorem 4.1.** (i) If  $A_0, A_1$  and  $B_0, B_1$  are two interpolation couples and  $S : A_0 + A_1 \rightarrow B_0 + B_1$  is a linear operator such that its restriction onto  $A_i$  is a continuous operator into  $B_i$  with operator norm  $C_i$  for  $i = 0, 1$ , then its restriction onto  $A_\theta = [A_0, A_1]_\theta$  is a continuous operator into  $B_\theta = [B_0, B_1]_\theta$  with operator norm  $C_0^{1-\theta} C_1^\theta$  for every  $\theta \in (0, 1)$ .

(ii) For a  $\sigma$ -finite measure space  $\mathfrak{M}$  and an interpolation couple of separable Banach spaces  $A_0, A_1$  we have

$$[L_{p_0}(\mathfrak{M}, A_0), L_{p_1}(\mathfrak{M}, A_1)]_\theta = L_p(\mathfrak{M}, [A_0, A_1]_\theta),$$

for every  $p_0, p_1 \in [1, \infty)$ ,  $\theta \in (0, 1)$ , where  $1/p = (1 - \theta)/p_0 + \theta/p_1$ .

(iii) Let  $H_p^m$  denote the Bessel potential spaces of complex-valued functions. Then for  $m_0, m_1 \in \mathbb{R}$  and  $1 < p_0, p_1 < \infty$

$$[H_{p_0}^{m_0}, H_{p_1}^{m_1}]_\theta = H_p^m,$$

where  $m = (1 - \theta)m_0 + \theta m_1$ , and  $1/p = (1 - \theta)/p_0 + \theta/p_1$ . Moreover, for integers  $m$  one has  $H_p^m = W_p^m$  with equivalent norms.

(iv) For a UMD Banach space  $V$ , denote by  $H_p^m(V)$  the Bessel potential spaces of  $V$ -valued functions. Then for  $1 < p < \infty$  and  $m_0, m_1 \in \mathbb{R}$

$$[H_p^{m_0}(V), H_p^{m_1}(V)]_\theta = H_p^m(V)$$

for every  $\theta \in (0, 1)$ , where  $m = (1 - \theta)m_0 + \theta m_1$ .

(v) For  $\theta \in [0, 1]$  there is a constant  $c_\theta$  such that

$$|v|_{A_\theta} \leq c_\theta |v|_{A_0}^{1-\theta} |v|_{A_1}^\theta$$

for all  $v \in A_0 \cap A_1$ .

5.  $L_p$  ESTIMATES

Let Assumptions 2.1 through 2.6 hold with an integer  $m \geq 0$ , and assume in this section that  $\mathcal{R} \equiv 0$ . Let  $u = (u_t)_{t \in [0, T]}$  be a  $W_p^{m+2}$ -valued solution to (1.1)-(1.2). Then for each  $\varphi \in C_0^\infty$  and any multi-number  $\alpha$  with  $|\alpha| \leq m$ , from the definition of the generalised solution and by integrating by parts we have

$$\begin{aligned} (D_\alpha u_t, \varphi) &= (D_\alpha \psi, \varphi) + \int_0^t (D_\alpha (\mathcal{A}_s u_s + f_s), \varphi) ds + \int_0^t (D_\alpha (\mathcal{M}_s^r u_s + g_s^r), \varphi) dw_s^r \\ &\quad + \int_0^t \int_Z (D_\alpha (I^\eta u_{s-} + \gamma_{s,z} T^\eta u_{s-} + h_s), \varphi) \tilde{\pi}(dz, ds) \end{aligned}$$

for all  $t \in [0, T]$  and almost all  $\omega \in \Omega$ . To shorten some expressions we introduce the operators  $\mathbb{J}^\xi$ ,  $\mathbb{J}^\eta$  and  $\mathbb{I}^\eta$ , defined by

$$\mathbb{J}^\xi \varphi = J^\xi \varphi + \lambda^\xi I^\xi \varphi, \quad \mathbb{J}^\eta \varphi = J^\eta \varphi + \lambda^\eta I^\eta \varphi, \quad \mathbb{I}^\eta \varphi = I^\eta \varphi + \gamma T^\eta \varphi$$

on functions  $\varphi$ . For functions  $v \in W_p^m$  let  $\mathbf{v} = \{v_\alpha : |\alpha| \leq m\}$  denote the vector whose coordinates are the derivatives  $v_\alpha := D_\alpha v$  for  $|\alpha| \leq m$ . We use also the notation  $I^\eta \mathbf{v}$ ,  $J^\eta \mathbf{v}$  and  $T^\eta \mathbf{v}$  for the vectors with coordinates  $I^\eta v_\alpha$ ,  $J^\eta v_\alpha$  and  $T^\eta v_\alpha$ , respectively for  $|\alpha| \leq m$ . Then applying Lemma 3.9 to  $\mathbf{u}_t$  we have

$$\begin{aligned} |\mathbf{u}_t|_{L_p}^p &= |\mathbf{u}_0|_{L_p}^p + p \int_0^t Q_p(s, u_s, f_s, g_s) + Q_p^\xi(s, u_s) + Q_p^\eta(s, u_s) ds \\ &\quad + p \int_0^t \int_{\mathbb{R}^d} |\mathbf{u}_s|^{p-2} D_\alpha u_s D_\alpha (\mathcal{M}_s^r u_s + g_s^r) dx dw_s^r \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} \{ (\sum_{|\alpha| \leq m} |D_\alpha u_{s-} + D_\alpha (\mathbb{I}^\eta u_{s-} + h_s)|^2)^{p/2} - |\mathbf{u}_{s-}|^p \\ &\quad - p \sum_{|\alpha| \leq m} |\mathbf{u}_{s-}|^{p-2} D_\alpha u_{s-} D_\alpha (\mathbb{I}^\eta u_{s-} + h_s) \} dx \pi(dz, ds) \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} p |\mathbf{u}_{s-}|^{p-2} \sum_{|\alpha| \leq m} D_\alpha u_{s-} D_\alpha (\mathbb{I}^\eta u_{s-} + h_s) dx \tilde{\pi}(dz, ds) \end{aligned} \quad (5.1)$$

almost surely for all  $t \in [0, T]$ , where we use the notation

$$\begin{aligned} Q_p(t, v, f, g) &= \int_{\mathbb{R}^d} p |\mathbf{v}|^{p-2} \{ \sum_{|\alpha| \leq m} v_\alpha D_\alpha (\mathcal{L}v + f) + \frac{1}{2} \sum_{r=1}^\infty \sum_{|\alpha| \leq m} |D_\alpha (\mathcal{M}^r v + g^r)|^2 \} dx \\ &\quad + \int_{\mathbb{R}^d} \frac{1}{2} p (p-2) |\mathbf{v}|^{p-4} \sum_{r=1}^\infty \sum_{|\alpha| \leq m} v_\alpha D_\alpha (\mathcal{M}^r v + g^r)^2 dx, \end{aligned} \quad (5.2)$$

$$Q_p^\xi(t, v) = \int_Z \int_{\mathbb{R}^d} p |\mathbf{v}|^{p-2} \sum_{|\alpha| \leq m} v_\alpha (\mathbb{J}^\xi v)_\alpha dx \nu(dz) \quad (5.3)$$

$$Q_p^\eta(t, v) = \int_Z \int_{\mathbb{R}^d} p |\mathbf{v}|^{p-2} \sum_{|\alpha| \leq m} v_\alpha (\mathbb{J}^\eta v)_\alpha dx \mu(dz)$$

for  $v \in W_p^{m+2}$ , for each  $f \in W_p^m$ ,  $g \in W_p^{m+1}(l_2)$ ,  $\omega \in \Omega$  and  $t \in [0, T]$ . In order to estimate  $E|\mathbf{u}_t|_{L_p}^p$  we define also the “ $p$ -form”

$$\begin{aligned} \hat{Q}_p(t, v, h) &= \int_Z \int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2} \sum_{|\alpha| \leq m} v_\alpha (\mathbb{J}^\eta v)_\alpha dx \mu(dz) \\ &+ \int_Z \int_{\mathbb{R}^d} \left| \sum_{|\alpha| \leq m} (v + \mathbb{I}^\eta v + h)_\alpha^2 \right|^{p/2} - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2} \sum_{|\alpha| \leq m} v_\alpha ((\mathbb{I}^\eta v)_\alpha + h_\alpha) dx \mu(dz) \end{aligned} \quad (5.4)$$

for  $v \in W_p^{m+2}$ ,  $h \in W_p^{m+2}(\mathcal{L}_{p,2})$ , for each  $\omega \in \Omega$  and  $t \in [0, T]$ .

**Proposition 5.1.** *Let Assumption 2.1 hold. Then for any  $p \in [2, \infty)$  there is a constant  $N = N(d, p, m, K)$  such that*

$$Q_p(v, t, f, g) \leq N(|v|_{W_p^m}^p + |f|_{W_p^m}^p + |g|_{W_p^{m+1}(l_2)}^p) \quad (5.5)$$

for all  $v \in W_p^{m+2}$ ,  $f \in W_p^m$ ,  $g \in W_p^{m+1}(l_2)$ ,  $\omega \in \Omega$  and  $t \in [0, T]$ .

*Proof.* This estimate is proved in [12] in a more general setting.  $\square$

**Proposition 5.2.** *Let Assumption 2.3 hold with an integer  $m \geq 0$ . Then for  $\omega \in \Omega$  and  $t \in [0, T]$*

$$\hat{Q}_p(v, t, h) \leq N(|v|_{W_p^m}^p + |h|_{W_p^{m+1}(\mathcal{L}_{p,2})}^p + \mathbf{1}_{p>2}|h|_{W_p^{m+2}(\mathcal{L}_{p,2})}^p) \quad (5.6)$$

for  $v \in W_p^{m+2}$  and  $h \in W_p^{m+2}(\mathcal{L}_{p,2})$  with constants  $N = N(d, m, p, K, K_{\bar{\eta}})$ .

*Proof.* First we claim that for any  $|\alpha| = n \leq m$

$$(\mathbb{I}^\eta v)_\alpha = I^\eta v_\alpha + \sum_{|\beta| \leq n} b^{\alpha\beta} T^\eta v_\beta + \sum_{|\beta| \leq n} c^{\alpha\beta} T^\eta v_\beta \quad (5.7)$$

$$(\mathbb{J}^\eta v)_\alpha = J^\eta v_\alpha + \sum_{|\beta| \leq n} b^{\alpha\beta} I^\eta v_\beta + \sum_{|\beta| \leq n} \bar{c}^{\alpha\beta} T^\eta v_\beta \quad (5.8)$$

with some functions  $b^{\alpha\beta}$ ,  $c^{\alpha\beta}$  and  $\bar{c}^{\alpha\beta}$  of  $(\omega, t, z, x)$ , such that for  $|\beta| \leq n$

$$|D^k b^{\alpha\beta}| \leq N\bar{\eta}, \quad |D^k c^{\alpha\beta}| \leq N\bar{\eta}^2, \quad |D^k \bar{c}^{\alpha\beta}| \leq N\bar{\eta}^2 \quad \text{for } k \leq m - n \quad (5.9)$$

with a constant  $N = N(d, m, K)$ . The reader can prove this easily by induction on  $n$ , noticing that

$$\begin{aligned} (J^\eta v)_i &= J^\eta v_i + \eta_i^k I^\eta v_k, & (I^\eta v)_i &= I^\eta v_i + \eta_i^k T^\eta v_k, \\ (T^\eta v)_i &= T^\eta v_i + \eta_i^k T^\eta v_k, \end{aligned}$$

where, as before, the subscripts indicates derivatives in the corresponding coordinates. Using equations (5.7) and (5.8) we have

$$\begin{aligned} \hat{Q}_p(t, v, h) &= \int_Z \int_{\mathbb{R}^d} |T^\eta \mathbf{v} + (\mathbf{b} + \mathbf{c})T^\eta \mathbf{v} + \mathbf{h}|^p - |\mathbf{v}|^p dx \mu(dz) \\ &- \int_Z \int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2} v_\alpha (I^\eta v_\alpha - J^\eta v_\alpha + b^{\alpha\beta} v_\beta + (c^{\alpha\beta} - \bar{c}^{\alpha\beta})T^\eta v_\beta + h_\alpha) dx \mu(dz), \end{aligned}$$

where  $\mathbf{b}$  and  $\mathbf{c}$  denote the matrices with entries  $b^{\alpha\beta}$  and  $c^{\alpha\beta}$ , respectively for  $|\alpha| \leq m$  and  $|\beta| \leq m$ . Clearly,

$$-p|\mathbf{v}|^{p-2}v_\alpha(I^\eta v_\alpha - J^\eta v_\alpha) = p|\mathbf{v}|^{p-2}v_\alpha D_i v_\alpha \eta^i = J^\eta |\mathbf{v}|^p - I^\eta |\mathbf{v}|^p,$$

and

$$|T^\eta \mathbf{v} + (\mathbf{b} + \mathbf{c})T^\eta \mathbf{v} + \mathbf{h}|^p - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2}v_\alpha(b^{\alpha\beta}v_\beta + h_\alpha) = \sum_{i=1}^5 A_i$$

with

$$\begin{aligned} A_1 &:= |T\mathbf{v}|^p - |\mathbf{v}|^p = I|\mathbf{v}|^p, \quad A_2 := p|T\mathbf{v}|^{p-2}T v_\alpha c^{\alpha\beta}T v_\beta \\ A_3 &:= |T\mathbf{v} + (\mathbf{b} + \mathbf{c})T\mathbf{v} + \mathbf{h}|^p - |T\mathbf{v}|^p - p|T\mathbf{v}|^{p-2}T v_\alpha((b^{\alpha\beta} + c^{\alpha\beta})T v_\beta + h_\alpha) \\ A_4 &:= pI(|\mathbf{v}|^{p-2}v_\alpha v_\beta)b^{\alpha\beta}, \quad A_5 := pI(|\mathbf{v}|^{p-2}v_\alpha)h_\alpha, \end{aligned}$$

where to ease notation we write  $I$  and  $T$  in place of  $I^\eta$  and  $T^\eta$ , respectively. Hence

$$\begin{aligned} \hat{Q}_p(v, t, h) &= \int_Z \int_{\mathbb{R}^d} J^\eta |\mathbf{v}|^p dx \mu(dz) + \sum_{i=2}^5 \int_Z \int_{\mathbb{R}^d} A_i dx \mu(dz) \\ &\quad + \int_Z \int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2}v_\alpha(\bar{c}^{\alpha\beta} - c^{\alpha\beta})T^\eta v_\beta dx \mu(dz). \end{aligned}$$

By Lemma 3.7

$$\int_Z \int_{\mathbb{R}^d} J^\eta |\mathbf{v}|^p dx \mu(dz) \leq N|v|_{W_p^m}^p,$$

and due to (5.9)

$$\int_Z \int_{\mathbb{R}^d} A_2 dx \mu(dz) \leq N|v|_{W_p^m}^p, \quad \int_Z \int_{\mathbb{R}^d} A_4 dx \mu(dz) \leq N|v|_{W_p^m}^p$$

with constants  $N = N(d, m, p, K, K_\eta)$ . By Taylor's formula, (5.9), and Assumption 2.3

$$\begin{aligned} A_3 &\leq p(p-1) \int_0^1 |T\mathbf{v} + \theta(\mathbf{b} + \mathbf{c})T\mathbf{v} + \theta\mathbf{h}|^{p-2} |(\mathbf{b} + \mathbf{c})T\mathbf{v} + \mathbf{h}|^2 d\theta \\ &\leq N(|\mathbf{h}|^p + T|\mathbf{v}|^{p-2}|\mathbf{h}|^2 + \eta^2 T|\mathbf{v}|^p). \end{aligned} \tag{5.10}$$

It is easy to see that

$$\int_Z \int_{\mathbb{R}^d} A_2 dx \mu(dz) \leq N|v|_{W_p^m}^p,$$

and

$$\int_Z \int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2}v_\alpha(\bar{c}^{\alpha\beta} - c^{\alpha\beta})T^\eta v_\beta dx \mu(dz) \leq N|v|_{W_p^m}^p.$$

Thus, from the above estimates we have

$$\begin{aligned} \hat{Q}_p(v, t, h) &\leq N(|v|_{W_p^m}^p + |h|_{W_p^m(\mathcal{L}_{p,2})}^p) + N \int_Z \int_{\mathbb{R}^d} T^\eta |\mathbf{v}|^{p-2} |\mathbf{h}|^2 dx \mu(dz) \\ &\quad + p \int_Z \int_{\mathbb{R}^d} I^\eta (|\mathbf{v}|^{p-2}v_\alpha) h_\alpha dx \mu(dz). \end{aligned}$$

It remains to estimate

$$H_p(v, h) := \int_Z \left| \int_{\mathbb{R}^d} I^\eta (|\mathbf{v}|^{p-2}v_\alpha) h_\alpha dx \right| \mu(dz),$$

$$K_p(v, h) := \int_Z \int_{\mathbb{R}^d} T^\eta |\mathbf{v}|^{p-2} |\mathbf{h}|^2 dx \mu(dz).$$

by the right-hand side of (5.6). If  $p = 2$ , then we have

$$K_2(v, h) = |\mathbf{h}|_{L_2(\mathcal{L}_2)}^2.$$

To estimate  $H_2(v, h)$  notice that by Taylor's formula, the change of variable  $y = \tau_{\theta\eta}(x) = x + \theta\eta(x)$  and by integration by parts we have

$$\begin{aligned} \int_{\mathbb{R}^d} I^\eta v_\alpha h_\alpha dx &= \int_{\mathbb{R}^d} \int_0^1 v_{\alpha i}(\tau_{\theta\eta}(x)) \eta^i(x) h_\alpha(x) d\theta dx \\ &= \int_{\mathbb{R}^d} \int_0^1 v_{\alpha i}(x) \eta^i(\tau_{\theta\eta}^{-1}(x)) h_\alpha(\tau_{\theta\eta}^{-1}(x)) |\det D\tau_{\theta\eta}^{-1}(x)| d\theta dx \\ &= \int_{\mathbb{R}^d} \int_0^1 v_\alpha(x) \zeta^{\alpha\beta}(x) h_{\alpha\beta}(\tau_{\theta\eta}^{-1}(x)) d\theta dx \end{aligned}$$

where  $\beta = 0, 1, 2, \dots, d$  and  $\zeta^{\alpha\beta}$  are some functions of  $(\omega, t, z, x, \theta)$  such that

$$\sum_{|\alpha| \leq m, |\beta| \leq 1} |\zeta^{\alpha\beta}|^2 \leq N \bar{\eta}^2$$

with a constant  $N = N(d, m, K)$ . Hence by the Cauchy-Schwarz and Young inequalities we get

$$\int_{\mathbb{R}^d} I^\eta v_\alpha h_\alpha dx \leq N \int_{\mathbb{R}^d} \bar{\eta}^2 |\mathbf{v}(x)|^2 dx + N' \int_{\mathbb{R}^d} \sum_{|\beta| \leq m+1} |h_\beta(x, z)|^2 dx$$

with  $N' = N'(d, m, K)$ , which gives the estimate for  $H_2(v, h)$  and finishes the proof of (5.6) for  $p = 2$ . If  $p > 2$  then by taking  $r = p$ ,  $\varphi = |\mathbf{v}|^{p-2} v_\alpha$  and  $h = h_\alpha$  for each  $\alpha$  in estimate (3.15) and using Young's inequality, we get

$$H_p(v, h) \leq N(|v|_{W_p^m}^p + |h|_{W_p^{m+2}(\mathcal{L}_{p,2})}^p).$$

Similarly, by taking  $r = p/2$ ,  $\phi = |\mathbf{v}|^{p-2}$  and  $g = |\mathbf{h}|^2$  in estimate (3.16) and using Young's inequality, we have

$$K_p(v, h) \leq N(|v|_{W_p^m}^p + |h|_{W_p^{m+2}(\mathcal{L}_{p,2})}^p),$$

which completes the proof of the proposition.  $\square$

**Corollary 5.3.** *Let Assumption 2.3 hold with an integer  $m \geq 0$ . Then for any  $p \geq 2$  and  $v \in W_p^{m+2}$*

$$Q_p^\eta(t, v) \leq N |v|_{W_p^m}^p \quad (5.11)$$

for all  $\omega \in \Omega$  and  $t \in [0, T]$  with a constant  $N = N(d, p, m, K, K_{\bar{\eta}})$ .

*Proof.* Notice that

$$Q_p^\eta(t, v) = \hat{Q}_p(t, v, 0) - \mathfrak{Q}_p(t, v),$$

where

$$\mathfrak{Q}_p(t, v) := \left| \sum_{|\alpha| \leq m} (v + \mathbb{I}^\eta v)_\alpha^2 \right|^{p/2} - |\mathbf{v}|^p - p |\mathbf{v}|^{p-2} \sum_{|\alpha| \leq m} v_\alpha (\mathbb{I}^\eta v)_\alpha.$$

By Proposition 5.2, we have

$$\hat{Q}_p(t, v, 0) \leq N|v|_{W_p^m}^p$$

with a constant  $N = N(d, p, m, K, K_{\bar{\eta}})$ . Moreover by the convexity of the function  $f(x) = |x|^p$ , we have  $\mathfrak{Q}_p(t, v) \geq 0$ , which finishes the poof of this corollary.  $\square$

For integers  $m \geq 0$  set

$$\mathfrak{P}_{m,p}^2(t, v, g) := \sum_{r=1}^{\infty} (p|\mathbf{v}|^{p-2}v_{\alpha}, (\mathcal{M}^r v + g^r)_{\alpha})^2 \quad (5.12)$$

$$\begin{aligned} \mathfrak{Q}_{m,p}(t, v, h) := & \int_Z \int_{\mathbb{R}^d} \{ (\sum_{|\alpha| \leq m} |(v + \mathbb{I}^{\eta} v + h)_{\alpha}|^2)^{p/2} - |\mathbf{v}|^p \\ & - p|\mathbf{v}|^{p-2}v_{\alpha}(\mathbb{I}^{\eta} v + h)_{\alpha} \} dx \mu(dz) \end{aligned} \quad (5.13)$$

for  $v \in W_p^{m+1}$ ,  $g \in W_p^{m+1}(l_2)$ ,  $h \in W_p^{m+1}(\mathcal{L}_{p,2})$ ,  $\omega \in \Omega$  and  $t \in [0, T]$ , where repeated indices  $\alpha$  mean summation over all multi-numbers of length  $m$ . (Recall that for functions  $v \in W_p^m$  we use the notation  $\mathbf{v}$  with coordinate  $v_{\alpha} = D_{\alpha}v$  for  $|\alpha| \leq m$ ).

**Proposition 5.4.** *Let  $m \geq 0$  be an integer and  $p \in [2, \infty)$ . Then the following estimates hold for all  $(\omega, t) \in \Omega \times [0, T]$ .*

(i) *If Assumption 2.1 is satisfied with  $m \geq 0$  then*

$$\mathfrak{P}_{m,p}^2(t, v, g) \leq N(|v|_{W_p^m}^{2p} + |v|_{W_p^m}^{2p-2}|g|_{W_p^m}^2) \quad (5.14)$$

*for all  $v \in W_p^{m+1}$  and  $g \in W_p^{m+1}(l_2)$ , with a constant  $N = N(d, m, p, K)$ .*

(ii) *If Assumption 2.3 is satisfied then*

$$\mathfrak{Q}_{m,p}(t, v, h) \leq N(|v|_{W_p^{m+1}}^p + |h|_{W_{p,2}^m}^p) \quad (5.15)$$

*for all  $v \in W_p^{m+1}$  and  $h \in W_p^m(\mathcal{L}_{p,2})$  with  $N = N(d, m, p, K, K_{\bar{\eta}})$ .*

*Proof.* Noticing that  $p|\mathbf{v}|^{p-2}v_{\alpha}\sigma^{ir}D_i v_{\alpha} = \sigma^{ir}D_i|\mathbf{v}|^p$ , by integration by parts and by Minkowski's and Hölder's inequalities we can see that

$$\bar{\mathfrak{P}}_{m,p}^2(t, v, g) := \sum_{r=1}^{\infty} (p|\mathbf{v}|^{p-2}v_{\alpha}, \mathcal{M}^r v_{\alpha} + g_{\alpha}^r)^2$$

can be estimated by the right-hand side of (5.14). By Minkowski and Hölder's inequalities it is easy to show that

$$\mathfrak{P}_{m,p}^2(t, v, g) - \bar{\mathfrak{P}}_{m,p}^2(t, v, g)$$

can also be estimated by the right-hand side of (5.14). To prove (ii) let  $\mathbf{y}$  denote the vector with coordinates  $y_{\alpha} = (\mathbb{I}^{\eta} v + h)_{\alpha}$  for  $|\alpha| \leq m$ . Then the integrand in (5.13) can be written as

$$A := |\mathbf{v} + \mathbf{y}|^p - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2}v_{\alpha}y_{\alpha}.$$

By Taylor's formula

$$0 \leq A \leq N(|\mathbf{v}|^{p-2}|\mathbf{y}|^2 + |\mathbf{y}|^p) \leq N'(|\mathbf{v}|^{p-2}|\mathbf{e}|^2 + |\mathbf{v}|^{p-2}|\mathbf{h}|^2 + |\mathbf{h}|^p + |\mathbf{e}|^p)$$

with constants  $N = N(d, p, m)$  and  $N' = N'(d, p, m)$ , where  $\mathbf{e}$  denotes the vector with coordinates  $e_\alpha := (\mathbb{I}^\eta v)_\alpha$  for  $|\alpha| \leq m$ . By Fubini's theorem and Hölder's inequality

$$\int_Z \int_{\mathbb{R}^d} |\mathbf{v}|^{p-2} |\mathbf{h}|^2 dx \mu(dz) = \int_{\mathbb{R}^d} |\mathbf{v}|^{p-2} |\mathbf{h}(x)|_{\mathcal{L}_2}^2 dx \leq |v|_{W_p^m}^{p-2} |h|_{W_p^m(\mathcal{L}_2)}^2.$$

Using Hölder's inequality, taking into account (5.7) and using Lemma 3.1 we obtain

$$\int_Z \int_{\mathbb{R}^d} |\mathbf{v}|^{p-2} |\mathbf{e}|^2 dx \mu(dz) \leq \int_Z |\mathbf{v}|_{L_p}^{p-2} |\mathbf{e}|_{L_p}^2 \mu(dz) \leq N^2 K_\eta^2 |v|_{W_p^m}^{p-2} |v|_{W_p^{m+1}}^2.$$

By Lemma 3.1 and (5.7) we have

$$\int_Z \int_{\mathbb{R}^d} |\mathbf{e}|^p dx \mu(dz) \leq N^p K^{p-2} K_\eta^2 |v|_{W_p^{m+1}}^p.$$

Combining these inequalities and using Young's inequality we get (5.15).  $\square$

## 6. PROOF OF THE MAIN RESULT

**6.1. Uniqueness of the generalised solution.** Let Assumptions 2.1 through 2.4 hold with  $m = 0$ . For a fixed  $p \in [2, \infty)$  let  $u^{(i)} = (u_t^{(i)})_{t \in [0, T]}$  be generalised solutions to equation (1.1) with initial condition  $u_0^{(i)} = \psi \in L_p$  for  $i = 1, 2$ . Then for  $v = u^{(2)} - u^{(1)}$  by Lemma 3.10 on Itô formula we have that almost surely

$$\begin{aligned} y_t := |v_t|_{L_p}^p &= \int_0^t Q_s(v_s) + \bar{Q}_s^\xi(v_s) + \bar{Q}_s^\eta(v_s) + (p|v_s|^{p-2} v_s, \mathcal{R}_s v_s) ds \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} P_s^\eta(z, v_{s-})(x) dx \pi(dz, ds) + \zeta_1(t) + \zeta_2(t) \end{aligned} \quad (6.1)$$

for all  $t \in [0, T]$ , where  $\zeta_1$  and  $\zeta_2$  are local martingales defined by

$$\begin{aligned} \zeta_1(t) &:= p \int_0^t \int_{\mathbb{R}^d} |v_s|^{p-2} v_s \mathcal{M}_s^r v_s dx dw_s^r, \\ \zeta_2(t) &:= p \int_0^t \int_Z \int_{\mathbb{R}^d} |v_{s-}|^{p-2} v_{s-} \mathbb{I}^\eta v_{s-} dx \tilde{\pi}(dz, ds), \end{aligned}$$

$Q_s(\cdot)$ ,  $\bar{Q}_s^\xi(\cdot)$ ,  $\bar{Q}_s^\eta(\cdot)$  and  $P_s^\eta(z, \cdot)$  are functionals on  $W_p^1$ , for each  $(\omega, s)$  and  $z$ , defined by

$$\begin{aligned} Q_s(v) &:= p \int_{\mathbb{R}^d} -D_i(|v|^{p-2} v) a_s^{ij} D_j v + \bar{b}_s^i |v|^{p-2} v D_i v + c_s |v|^p + \frac{p-1}{2} |v|^{p-2} \sum_{r=1}^\infty |\mathcal{M}_s^r v|^2 dx, \\ \bar{Q}_s^\xi(v) &= p \int_{\mathbb{R}^d} -D_i(|v|^{p-2} v) \mathcal{J}_\xi^i v + |v|^{p-2} v \mathcal{J}_\xi^0 v dx + \int_Z \int_{\mathbb{R}^d} p |v|^{p-2} v \lambda_{s,z}^\xi I^\xi v dx dz, \\ \bar{Q}_s^\eta(v) &= p \int_{\mathbb{R}^d} -D_i(|v|^{p-2} v) \mathcal{J}_\eta^i v + |v|^{p-2} v \mathcal{J}_\eta^0 v dx + \int_Z \int_{\mathbb{R}^d} p |v|^{p-2} v \lambda_{s,z}^\eta I^\eta v dx dz, \\ P_s^\eta(z, v) &:= |v + \mathbb{I}^\eta v|^p - |v|^p - p |v|^{p-2} v \mathbb{I}^\eta v. \end{aligned}$$

Recall that  $\bar{b}^i = b - D_j a^{ij}$ ,  $\mathcal{J}_\eta^i$  and  $\mathcal{J}_\eta^0$  are defined by (2.3)-(2.4), and  $(v, w)$  denotes the Lebesgue integral over  $\mathbb{R}^d$  of the product  $vw$  for real functions  $v$  and  $w$  on  $\mathbb{R}^d$ .



Note that due to the convexity of the function  $|r|^p$ ,  $r \in \mathbb{R}$ , we have

$$P_s^\eta(z, v)(x) \geq 0 \quad \text{for all } (\omega, s, z, x) \quad (6.2)$$

for real-valued functions  $v = v(x)$ ,  $x \in \mathbb{R}^d$ . Together with the above functionals we need also estimate the functionals  $\mathfrak{Q}_s(\cdot)$  and  $\hat{Q}_s^\eta(\cdot)$  defined for each  $(\omega, s) \in \Omega \times [0, T]$  by

$$\mathfrak{Q}_s(v) := \int_Z \int_{\mathbb{R}^d} P_s^\eta(z, v)(x) dx \mu(dz), \quad \hat{Q}_s^\eta(v) := \mathfrak{Q}_s(v) + \bar{Q}_s^\eta(v)$$

for  $v \in W_p^1$ .

**Proposition 6.1.** *Let Assumptions 2.1, 2.2 and 2.3 hold with  $m = 0$ . Then for  $p \geq 2$  there are constants  $N = N(d, p, K)$ ,  $N_1 = N_2(d, p, K, K_{\bar{\eta}})$  and  $N_2 = N_2(d, p, K, K_{\bar{\eta}})$  such that*

$$Q_s(v) \leq N|v|_{L_p}^p, \quad \bar{Q}_s^\xi(v) \leq N_1|v|_{L_p}^p, \quad \bar{Q}_s^\eta(v) \leq N_2|v|_{L_p}^p, \quad \hat{Q}_s^\eta(v) \leq N_2|v|_{L_p}^p, \quad (6.3)$$

$$\mathfrak{Q}(v) \leq N_2|v|_{W_p^1}^p \quad (6.4)$$

for all  $v \in W_p^1$  and  $(\omega, s) \in \Omega \times [0, T]$ .

*Proof.* Notice that the estimate (6.4) is the special case of Proposition 5.4 (ii), and for  $v \in W_p^2$  the second and third estimates in (6.3) follow from the estimate (5.11) in Corollary 5.3. Notice also that for  $v \in W_p^2$  the first estimate in (6.3) is a special case of (5.5) in Proposition 5.1, and the last estimate in (6.3) is a special case of the estimate in Proposition 5.2. It is an easy exercise to show that the functionals on the left-hand side of the inequalities in (6.3) are continuous in  $v \in W_p^1$ , that completes the proof of the proposition.  $\square$

Define now the stochastic process

$$X_t := |v_t|_{L_p}^p + \int_0^t |v_s|_{W_p^1}^p ds, \quad t \in [0, T]$$

and the stopping time

$$\tau_n := \inf\{t \in [0, T] : X_t \geq n\} \wedge \rho_n$$

for every integer  $n \geq 1$ , where  $(\rho_n)_{n=1}^\infty$  is an increasing sequence of stopping times, converging to infinity such that  $(\zeta_i(t \wedge \rho_n))_{t \in [0, T]}$  is a martingale for each  $n \geq 1$  and  $i = 1, 2$ . Then clearly,  $E\zeta_i(t \wedge \tau_n) = 0$  for  $t \in [0, T]$  and  $i = 1, 2$ . Due to (6.2) and the estimate in (6.4) we have

$$E \int_0^{T \wedge \tau_n} \int_Z \int_{\mathbb{R}^d} |P_s^\eta(z, v_{s-})(x)| dx \mu(dz) ds \leq NE \int_0^{T \wedge \tau_n} |v_s|_{W_p^1}^p ds < \infty,$$

which implies

$$\begin{aligned} & E \int_0^{t \wedge \tau_n} \int_Z \int_{\mathbb{R}^d} P_s^\eta(z, v_{s-})(x) dx \pi(dz, ds) \\ &= E \int_0^{t \wedge \tau_n} \int_Z \int_{\mathbb{R}^d} P_s^\eta(z, v_{s-})(x) dx \mu(dz) ds = E \int_0^{t \wedge \tau_n} \mathfrak{Q}_s(v_{s-}) ds. \end{aligned}$$

Thus, substituting  $t \wedge \tau_n$  in place of  $t$  in (6.1) and then taking expectation and using Proposition 6.1 and Assumption 2.5 we obtain

$$Ey_{t \wedge \tau_n} = E \int_0^{t \wedge \tau_n} Q_s(v_s) + \bar{Q}_s^\xi(v_s) + \hat{Q}_s^\eta(v_s) + (p|v_s|^{p-2}v_s, \mathcal{R}v_s) ds$$

$$\leq NE \int_0^{t \wedge \tau_n} |v_s|_{L_p}^p ds \leq N \int_0^t Ey(s \wedge \tau_n) ds \leq NTn < \infty$$

for  $t \in [0, T]$ . Hence by Gronwall's lemma  $Ey(t \wedge \tau_n) = 0$  for each  $t \in [0, T]$  and integer  $n \geq 1$ , which implies that almost surely  $y_t = 0$  for all  $t \in [0, T]$ , and completes the proof of the uniqueness.

## 6.2. A priori estimates.

**Proposition 6.2.** *Let Assumptions 2.1 through 2.4 and assume Assumption 2.6 hold with an integer  $m \geq 0$  and  $\mathcal{R} \equiv 0$ . Let  $u = (u_t)_{t \in [0, T]}$  be a  $W_p^{m+2}$ -valued generalised solution to (1.1)-(1.2) such that it is cadlag as a  $W_p^m$ -valued process and*

$$E \int_0^T |u_t|_{W_p^{m+2}}^p dt + E \sup_{t \leq T} |u_t|_{W_p^m}^p < \infty.$$

Then

$$E \sup_{t \leq T} |u_t|_{W_p^n}^p \leq N(E|\psi|_{W_p^n}^p + EK_{n,p}^p(T)) \quad \text{for every integer } n \in [0, m] \quad (6.5)$$

with a constant  $N = N(m, d, p, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ .

*Proof.* We may assume that the right-hand side of the inequality (6.5) is finite. For a fixed integer  $n \in [0, m]$ , multi-numbers  $|\alpha| \leq n$  and  $\varphi \in C_0^\infty$ , we have

$$\begin{aligned} d(D_\alpha u_t, \varphi) &= (D_\alpha \mathcal{A}_t u_t + D_\alpha f_t, \varphi) dt + (D_\alpha \mathcal{M}_t^r u_t + D_\alpha g_t^r, \varphi) dw_t^r \\ &\quad + \int_Z (D_\alpha (\mathbb{I}^\eta u_{t-} + h_t(z)), \varphi) \tilde{\pi}(dz, dt). \end{aligned}$$

For an integer  $n \leq m$  let  $\mathbf{u}$  denote the vector with coordinates  $u_\alpha := D_\alpha u$  for  $|\alpha| \leq n$ . Recall, see (5.1), that by Lemma 3.9 on Itô's formula we have

$$d|\mathbf{u}_t|_{L_p}^p = (Q_p(t, u_t, f_t, g_t) + Q_p^\xi(t, u_t) + \hat{Q}_p(t, u_t, h_t)) dt + \sum_{i=1}^3 d\zeta_i(t),$$

where the  $Q_p$ ,  $Q_p^\xi$  and  $\hat{Q}_p$  are defined in (5.2), (5.3) and (5.4), with  $n$  in place of  $m$ , and  $\zeta_i = (\zeta_i(t))_{t \in [0, T]}$  is a cadlag local martingale starting from zero for each  $i = 1, 2, 3$ , such that

$$\begin{aligned} d\zeta_1(t) &= p(|\mathbf{u}_t|^{p-2} D_\alpha u_t, D_\alpha \mathcal{M}_t^r u_t + D_\alpha g_t^r) dw_t^r, \\ d\zeta_2(t) &:= p \int_Z (|\mathbf{u}_{t-}|^{p-2} D_\alpha u_{t-}, D_\alpha \mathbb{I}^\eta u_{t-} + D_\alpha h_{t,z}) \tilde{\pi}(dz, dt) \end{aligned} \quad (6.6)$$

and

$$d\zeta_3(t) := \int_Z P_p(t, u_{t-}, h_t) \pi(dz, dt) - \int_Z P_p(t, u_{t-}, h_t) \mu(dz) dt, \quad (6.7)$$

where

$$P_p(t, v, h) := \int_{\mathbb{R}^d} \left| \sum_{|\alpha| \leq n} |(v + \mathbb{I}^\eta v + h)_\alpha|^2 \right|^{p/2} - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2} v_\alpha (\mathbb{I}^\eta v + h)_\alpha dx$$

for  $v \in W_p^{m+2}$  and  $h \in W_{p,2}^{m+2}$ . By Proposition 5.1 and Corollary 5.3 we obtain

$$d|\mathbf{u}_t|^p \leq N(|u_t|_{W_p^n}^p dt + d\mathcal{K}_{n,p}^p(t)) + \sum_{i=1}^3 d\zeta_i(t).$$

Hence using the estimate (5.15) in Proposition 5.4 we have

$$E|\mathbf{u}_{t \wedge \tau_k}|^p \leq E|\mathbf{u}_0|^p + N \int_0^t E|u_{s \wedge \tau_k}|_{W_p^n}^p ds + NE\mathcal{K}_{n,p}^p(T \wedge \tau_k)$$

for all  $t \in [0, T]$ , for a localising sequence  $(\tau_k)_{k=1}^\infty$  of stopping times for  $\zeta_i$ ,  $i = 1, 2, 3$ . Hence by Gronwall's lemma

$$E|u_{t \wedge \tau_k}|_{W_p^n}^p \leq N(E|\psi|_{W_p^n}^p + E\mathcal{K}_{n,p}^p(T))$$

for  $t \in [0, T]$  and  $k \geq 1$  with a constant  $N = N(d, m, p, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ , which implies

$$\sup_{t \leq T} E|u_t|_{W_p^n}^p \leq N(E|\psi|_{W_p^n}^p + E\mathcal{K}_{n,p}^p(T)) \quad (6.8)$$

by Fatou's lemma. To show that we can interchange the supremum and expectation it suffices to prove that for every  $\varepsilon > 0$

$$E \sup_{t \leq T} |\zeta_1(t)| \leq \varepsilon E \sup_{t \leq T} |u_t|_{W_p^n}^p + N(E|\psi|_{W_p^n}^p + E\mathcal{K}_{n,p}^p(T)) < \infty \quad (6.9)$$

and

$$E \sup_{t \leq T} |\zeta_2(t) + \zeta_3(t)| \leq \varepsilon E \sup_{t \leq T} |u_t|_{W_p^n}^p + N(E|\psi|_{W_p^n}^p + E\mathcal{K}_{n,p}^p(T)) < \infty \quad (6.10)$$

with a constant  $N = N(\varepsilon, d, m, p, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ . The proof of (6.9) is well-known and it goes as follows. Recall the notation  $\mathfrak{P}_p^2$  in (5.12) and notice that by the Davis inequality, using the estimate in (5.14) we have

$$\begin{aligned} E \sup_{t \leq T} |\zeta_1(t)| &\leq 3E \left( \int_0^T \mathfrak{P}_p^2(t, u_t, g_t) dt \right)^{1/2} \\ &\leq NE \left( \int_0^T |u_t|_{W_p^n}^{2p} + |u_t|_{W_p^n}^{2p-2} |g_t|_{W_p^n}^2 dt \right)^{1/2} \\ &\leq NE \left( \sup_{t \leq T} |u_t|_{W_p^n}^p \int_0^T |u_t|_{W_p^n}^p + |u_t|_{W_p^n}^{p-2} |g_t|_{W_p^n}^2 dt \right)^{1/2} \\ &\leq \varepsilon E \sup_{t \leq T} |u_t|_{W_p^n}^p + \varepsilon^{-1} N^2 E \int_0^T |u_t|_{W_p^n}^p + |g_t|_{W_p^n}^p dt < \infty, \end{aligned} \quad (6.11)$$

which gives (6.9) by virtue of (6.8). To prove (6.10) we first assume that  $\mu$  is a finite measure. Notice that

$$\zeta(t) := \zeta_2(t) + \zeta_3(t) = \int_0^t \int_Z A(s, u_s, h_s) \tilde{\pi}(dz) ds,$$

where  $A(s, v, h)$  is defined by

$$A(s, v, h) := \left| \sum_{|\alpha| \leq n} (v + \mathbb{I}^\eta v + h)_\alpha^2 \right|^{p/2} - |\mathbf{v}|^p$$

for  $v \in W_p^m$ ,  $\mathbf{v} = (v_\alpha)_{|\alpha| \leq n}$  and  $h \in W_p^{m+2}(\mathcal{L}_{p,2})$ . By similar calculations to those in the proof of Proposition 5.2 we can easily see that

$$A(s, v, h) = \sum_{i=1}^6 B_i(s, v, h)$$

with

$$\begin{aligned} B_1 &= I^\eta |\mathbf{v}|^p, \quad B_2 = p |\mathbf{v}|^{p-2} v_\alpha h_\alpha, \\ B_3 &:= |T^\eta \mathbf{v} + (\mathbf{b} + \mathbf{c}) T^\eta \mathbf{v} + \mathbf{h}|^p - |T^\eta \mathbf{v}|^p - p |T^\eta \mathbf{v}|^{p-2} T^\eta v_\alpha ((b^{\alpha\beta} + c^{\alpha\beta}) T^\eta v_\beta + h_\alpha), \\ B_4 &= p I^\eta (|\mathbf{v}|^{p-2} v_\alpha) h_\alpha \\ B_5 &= p |T^\eta \mathbf{v}|^{p-2} T^\eta v_\alpha b^{\alpha\beta} T v_\beta, \quad B_6 = p |T^\eta \mathbf{v}|^{p-2} T^\eta v_\alpha c^{\alpha\beta} T^\eta v_\beta, \end{aligned}$$

where  $\mathbf{b} = (b^{\alpha\beta})$  and  $\mathbf{c} = (c^{\alpha\beta})$  are from (5.7). Hence

$$\zeta(t) = \sum_{i=1}^6 (\rho_{i1}(t) - \rho_{i2}(t))$$

with

$$\begin{aligned} \rho_{i1}(t) &= \int_0^t \int_Z \int_{\mathbb{R}^d} B_i(s, u_{s-}, h) \pi(dz, ds), \\ \rho_{i2}(t) &= \int_0^t \int_Z \int_{\mathbb{R}^d} B_i(s, u_{s-}, h) dx \mu(dz) ds. \end{aligned}$$

Note that for  $\rho_i(t) := \rho_{i1}(t) - \rho_{i2}(t)$  one can always have the supremum estimate

$$\begin{aligned} E \sup_{t \leq T} |\rho_i(t)| &\leq E \sup_{t \leq T} |\rho_{i1}(t)| + E \sup_{t \leq T} |\rho_{i2}(t)| \\ &\leq 2E \int_0^T \int_Z \left| \int_{\mathbb{R}^d} B_i(t, u_s, h) dx \right| \mu(dz) dt. \end{aligned} \tag{6.12}$$

This, however, is not always useful, and when almost surely

$$\langle \rho_i \rangle(T) = \int_0^T \int_Z \left| \int_{\mathbb{R}^d} B_i(s, u_{s-}, h) dx \right|^2 \mu(dz) ds < \infty,$$

then we can view  $\rho_i(t)$  as the stochastic Itô integral

$$\int_0^t \int_Z \int_{\mathbb{R}^d} B_i(s, u_{s-}, h) \tilde{\pi}(dz, ds)$$

and apply the Davis inequality

$$E \sup_{t \leq T} |\rho_i(t)| \leq 3E \langle \rho_i \rangle^{1/2}(T).$$

By Minkowski's and Hölder's inequalities

$$\begin{aligned} \int_0^T \int_Z \left| \int_{\mathbb{R}^d} |\mathbf{u}_s|^{p-2} D_\alpha u_s D_\alpha h_s dx \right|^2 \mu(dz) ds &\leq \int_0^T \left( \int_{\mathbb{R}^d} |\mathbf{u}_s|^{p-1} |\mathbf{h}_s|_{\mathcal{L}_2} dx \right)^2 ds \\ &\leq \int_0^T |u_s|_{W_p^{n,2}}^{2p-2} |h_s|_{W_{p,2}^n}^2 ds < \infty. \end{aligned}$$

Thus we can view  $\zeta_2(t)$  as a stochastic Itô integral, and applying the Davis inequality we get

$$E \sup_{t \leq T} |\zeta_i(t)| \leq \varepsilon E \sup_{t \leq T} |u_t|_{W_p^n}^p + \varepsilon^{-1} N(E|\psi|_{W_p^n}^p + EK_{n,p}^p(T)) \quad (6.13)$$

for  $i = 2$  in the same way as estimate in (6.9) is proved. By Lemma 3.7 (iii) we have

$$\int_0^T \int_Z \left| \int_{\mathbb{R}^d} I^\eta |\mathbf{u}_s|^p dx \right|^2 \mu(dz) ds \leq N \int_0^T |u_s|_{W_p^n}^{2p} ds,$$

which, as before, allows us to get the estimate (6.13) for  $i = 1$ . By estimate (3.15), Young's inequality and (6.8), we have

$$\begin{aligned} & E \int_0^T \int_Z \left| \int_{\mathbb{R}^d} I^\eta (|\mathbf{u}_{s-}|^{p-2} D_\alpha u_{s-}) D_\alpha h_s dx \right| \mu(dz) ds \\ & \leq NE \int_0^T |u_s|_{W_p^n}^p ds + NEK_{n,p}^p(T) \leq N'(E|\psi|_{W_p^n}^p + EK_{n,p}^p(T)) \end{aligned}$$

with constants  $N$  and  $N'$  depending only on  $K, d, m, p, T, K_{\bar{\varepsilon}}$  and  $K_{\bar{\eta}}$ . Thus we can use estimate (6.12) to get

$$E \sup_{t \leq T} |\zeta_i(t)| \leq NE|\psi|_{W_p^n}^p + NEK_{n,p}^p(T) \quad (6.14)$$

for  $i = 4$  with a constant  $N = N(K, d, m, p, T, K_{\bar{\varepsilon}}, K_{\bar{\eta}})$ . Similarly, using the estimate for  $A_3$  in (5.10) and the estimate (3.16) we obtain (6.14) for  $i = 3$ . Due to (5.9)

$$\left| \int_{\mathbb{R}^d} B_5(t, v, h) dx \right| \leq N\bar{\eta}|v|_{W_p^n}^p, \quad \left| \int_{\mathbb{R}^d} B_6(t, v, h) dx \right| \leq N\bar{\eta}^2|v|_{W_p^n}^p$$

for  $v \in W_p^m$ . Consequently, viewing  $\zeta_5(t)$  as Itô integral we get the estimate (6.13) for  $i=5$ , and applying estimate (6.12) to  $\zeta_6(t)$  we obtain the estimate (6.14) for  $i = 6$ . Combining the estimates (6.13) for  $i = 1, 2, 5$  and the estimate (6.13) for  $i = 3, 4, 6$  we obtain (6.11).

In the general case of  $\sigma$ -finite measure  $\mu$  we have a nested sequence  $(Z_k)_{k=1}^\infty$  of sets  $Z_k \in \mathcal{Z}$  such that  $\mu(Z_k) < \infty$  for every  $k$  and  $\cup_{k=1}^\infty Z_k = Z$ . For each integer  $k \geq 1$  define the measures

$$\pi_k(F) = \pi((Z_k \times (0, T]) \cap F), \quad \mu_k(G) = \mu(Z_k \cap G)$$

for  $F \in \mathcal{Z} \otimes \mathcal{B}((0, T])$  and  $G \in \mathcal{Z}$ , and set  $\tilde{\pi}_k(dz, dt) = \pi_k(dz, dt) - \mu_k \otimes dt$ . Let  $\zeta_2^{(k)}$  and  $\zeta_3^{(k)}$  be defined as  $\zeta_2$  and  $\zeta_3$ , respectively, but with  $\tilde{\pi}_k, \pi_k$  and  $\mu_k$  in place of  $\tilde{\pi}, \pi$  and  $\mu$ , respectively, in (6.6) and (6.7). By virtue of what we have proved above, for each  $k$  we have

$$E \sup_{t \leq T} |\zeta_2^{(k)}(t) + \zeta_3^{(k)}(t)| \leq \varepsilon E \sup_{t \leq T} |u_t|_{W_p^n}^p + N(E|\psi|_{W_p^n}^p + EK_{n,p}^p(T)) < \infty \quad (6.15)$$

for  $\varepsilon > 0$  with a constant  $N = N(\varepsilon, m, p, T, K, K_{\bar{\varepsilon}}, K_{\bar{\eta}})$ . Note that for a subsequence  $k' \rightarrow \infty$

$$\zeta_i^{(k')}(t) \rightarrow \zeta_i(t) \quad \text{almost surely, uniformly in } t \in [0, T]$$

for  $i = 2, 3$ . Hence letting  $k = k' \rightarrow \infty$  in (6.15) by Fatou's lemma we obtain (6.10), which completes the proof of the proposition.  $\square$

**6.3. Existence of a generalised solution.** Before the construction of a generalised solution to (1.1)-(1.2), we introduce some notations. For integers  $r > 1$ , real numbers  $n \geq 0$  and  $p \geq 2$  let  $\mathbb{U}_{r,p}^n$  denote the space of  $H_p^n$ -valued  $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable functions  $v$  on  $\Omega \times [0, T]$  such that

$$|v|_{\mathbb{U}_{r,p}^n}^p := E \left( \int_0^T |v_t|_{H_p^n}^r dt \right)^{p/r} < \infty.$$

The subspace of well-measurable functions  $v : \Omega \times [0, T] \rightarrow H_p^n$  in  $\mathbb{U}_{r,p}^n$  is denoted by  $\mathbb{V}_{r,p}^n$ , and we will use  $\mathbb{V}_p^n$  to denote  $\mathbb{V}_{p,p}^n$ . Set  $\Psi_p^n := L_p(\Omega, \mathcal{F}_0; H_p^n)$ , and recall from the Introduction the definition of the spaces  $\mathbb{H}_p^n = \mathbb{H}_p^n(\mathbb{R})$  and  $\mathbb{H}_p^n(V)$  for separable Banach spaces  $V$ .

In the whole section we assume that Assumptions 2.1 through 2.6 with  $m \geq 1$  are in force. By a standard stopping time argument we may assume that

$$E|\psi|_{H_p^m}^p + EK_{p,m}^p(T) < \infty. \quad (6.16)$$

First we assume that  $m$  is an integer,  $\mathcal{R} \equiv 0$ , and make also the following additional assumption

**Assumption 6.1.** The initial condition  $\psi$  and the free data  $f$ ,  $g$  and  $h$  vanish if  $|x| \geq R$  for some  $R > 0$ .

Under the above conditions we approximate the Cauchy problem (1.1)-(1.2) by mollifying (in  $x \in \mathbb{R}^d$ ) all data and coefficients involved in it. For  $\varepsilon \in (0, \varepsilon_0)$  we consider the equation

$$\begin{aligned} dv_t(x) &= \left( \mathcal{A}_t^\varepsilon v_t(x) + f_t^{(\varepsilon)}(x) \right) dt + \left( \mathcal{M}_t^{\varepsilon r} v_t(x) + g_t^{(\varepsilon)r}(x) \right) dw_t^r \\ &+ \int_Z \left( I^{\eta^{(\varepsilon)}} v_{t-}(x) + \gamma_{t,z}^{(\varepsilon)}(x) I^{\eta^{(\varepsilon)}} v_{t-}(x) + h_t^{(\varepsilon)}(x, z) \right) \tilde{\pi}(dz, dt), \end{aligned} \quad (6.17)$$

with initial condition

$$v_0(x) = \psi^{(\varepsilon)}, \quad (6.18)$$

where  $\varepsilon_0$  is given in Corollary 3.4,

$$\mathcal{M}^{\varepsilon r} = \sigma^{(\varepsilon)ir} D_i + \beta^{(\varepsilon)r}, \quad \mathcal{A}^\varepsilon = \mathcal{L}^\varepsilon + \mathcal{N}^{\xi^{(\varepsilon)}} + \mathcal{N}^{\eta^{(\varepsilon)}}$$

with operators

$$\mathcal{L}^\varepsilon = a^{\varepsilon ij} D_{ij} + b^{(\varepsilon)i} D_i + c^{(\varepsilon)}, \quad a^\varepsilon = a^{(\varepsilon)} + \varepsilon \mathbf{I},$$

and

$$\begin{aligned} \mathcal{N}_t^{\xi^{(\varepsilon)}} \varphi(x) &= \int_Z J^{\xi^{(\varepsilon)}} \varphi(x) + (\lambda_{t,z}^{\xi^{(\varepsilon)}}(x) I^{\xi^{(\varepsilon)}} \varphi(x) \nu(dz), \\ \mathcal{N}_t^{\eta^{(\varepsilon)}} \varphi(x) &= \int_Z J^{\eta^{(\varepsilon)}} \varphi(x) + (\lambda_{t,z}^{\eta^{(\varepsilon)}}(x) I^{\eta^{(\varepsilon)}} \varphi(x) \mu(dz) \end{aligned}$$

for real-valued differentiable functions  $\varphi$ . Recall that  $\mathbf{I}$  denotes the identity matrix, and  $v^{(\varepsilon)}$  denotes the mollification  $v^{(\varepsilon)} = S^\varepsilon v$  of  $v$  in  $x \in \mathbb{R}^d$  defined in (3.4).

Note that by virtue of standard properties of mollifications and by Corollary 3.4 and Lemma 3.5, we have that Assumptions 2.1 through 2.6 are satisfied for (6.17)-(6.18) with every integer  $m \geq 0$  with non-negative functions  $\bar{\xi} = \bar{\xi}_m(z)$ ,  $\bar{\eta} = \bar{\eta}_m(z)$  of  $z \in Z$  and constants  $K = K_m$ ,

$$K_\xi^2 := K_{\xi,m}^2 = \int_Z \bar{\xi}_m^2(z) \nu(dz) < \infty, \quad K_\eta^2 := K_{\eta,m}^2 = \int_Z \bar{\eta}_m^2(z) \mu(dz) < \infty.$$

Moreover, there is a constant  $\delta > 0$  such that  $P \otimes dt \otimes dx$ -almost all  $(\omega, t, x) \in \Omega \times H_T$

$$(2a^{\varepsilon ij} - \sigma^{(\varepsilon)ir} \sigma^{(\varepsilon)jr}) z^i z^j \geq \delta |z|^2 \quad \text{for all } z = (z^1, \dots, z^d) \in \mathbb{R}^d.$$

Due to (6.16) and Assumption 6.1

$$E|\psi|_{W_2^n}^2 + EK_{n,2}^2(T) < \infty$$

for each  $n$ . Hence by [15] the Cauchy problem (6.17)-(6.18) has a unique generalised solution  $u^\varepsilon$ , which is a  $W_2^n$ -valued cadlag process and for each integer  $n \geq 0$ , and there is a constant  $N$  such that

$$E \sup_{t \leq T} |u_t^\varepsilon|_{W_2^n}^2 \leq N(E|\psi|_{W_2^n}^2 + EK_{n,2}^2(T)) < \infty.$$

Thus by Sobolev's embedding  $u^\varepsilon$  is a cadlag  $W_p^n$ -valued process for every  $n$  such that

$$E \sup_{t \leq T} |u_t^\varepsilon|_{W_p^n}^p < \infty.$$

Moreover, by Proposition 6.2 and Lemma 3.5 for  $m \geq 1$  and  $n = 0, 1, \dots, m$  we have

$$|u^\varepsilon|_{\mathbb{V}_{r,p}^n} \leq N(|\psi|_{\Psi_p^n} + |f|_{\mathbb{H}_p^n} + |g|_{\mathbb{H}_p^{n+1}(l_2)} + |h|_{\mathbb{H}_p^{n+i}(\mathcal{L}_{p,2})}) \quad \text{for } n = 0, 1, 2, \dots, m \quad (6.19)$$

for every integer  $r > 1$  with a constant  $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ , where  $i = 1$  when  $p = 2$  and  $i = 2$  when  $p > 2$ . Recall that  $\mathbb{V}_{r,p}^n$  denotes the subspace of well-measurable functions  $v : \Omega \times [0, T] \rightarrow H_p^n$  in  $\mathbb{U}_{r,p}^n$ . Since  $\mathbb{V}_{r,p}^n$  is reflexive, there exists a sequence  $\{\varepsilon_k\}_{k=1}^\infty$  and a process  $u \in \mathbb{V}_{r,p}^n$  such that  $\lim_{k \rightarrow \infty} \varepsilon_k = 0$  and  $u^{\varepsilon_k}$  converges weakly to some  $u$  in  $\mathbb{V}_{r,p}^n$ . To show that a modification of  $u$  is a solution to (1.1)-(1.2) we pass to the limit in the equation

$$\begin{aligned} (u_t^\varepsilon, \varphi) &= (\psi^{(\varepsilon)}, \varphi) + \int_0^t \langle \mathcal{A}_s^\varepsilon u_s^\varepsilon, \varphi \rangle + (f_s^{(\varepsilon)}, \varphi) ds + \int_0^t (\mathcal{M}_s^{\varepsilon r} u_s^\varepsilon + g_s^{(\varepsilon)r}, \varphi) dw_s^r \\ &\quad + \int_0^t \int_Z (I^{\eta^{(\varepsilon)}} u_s^\varepsilon + \gamma_{s,z}^{(\varepsilon)} T^{\eta^{(\varepsilon)}} u_s^\varepsilon + h_s^{(\varepsilon)}, \varphi) \tilde{\pi}(dz, ds) \end{aligned} \quad (6.20)$$

where  $\varphi \in C_0^\infty$ , and  $\langle \mathcal{A}_s^\varepsilon u_s^\varepsilon, \varphi \rangle$  is defined as  $\langle \mathcal{A}_s u_s, \varphi \rangle$  in (2.5) but with  $u^\varepsilon$ ,  $a^\varepsilon$ ,  $b^{(\varepsilon)}$ ,  $c^{(\varepsilon)}$ ,  $\xi^{(\varepsilon)}$ ,  $\eta^{(\varepsilon)}$ ,  $(\lambda^\xi)^{(\varepsilon)}$  and  $(\lambda^\eta)^{(\varepsilon)}$  in place of  $u$ ,  $a$ ,  $b$ ,  $c$ ,  $\xi$ ,  $\eta$ ,  $\lambda^\xi$  and  $\lambda^\eta$  respectively. To this end we take a bounded well-measurable real-valued process  $\zeta = (\zeta_t)_{t \in [0, T]}$ , multiply both sides of equation (6.20) with  $\zeta_t$  and then integrate the expressions we get against  $P \otimes dt$  over  $\Omega \times [0, T]$ . Thus we obtain

$$\begin{aligned} F(u^\varepsilon) &= E \int_0^T \zeta_t (\psi^{(\varepsilon)}, \varphi) dt + \sum_{i=1}^3 F_\varepsilon^i(u^\varepsilon) + E \int_0^T \int_0^t \zeta_t (f_s^{(\varepsilon)}, \varphi) ds dt \\ &\quad + E \int_0^T \zeta_t \int_0^t (g_s^{(\varepsilon)r}, \varphi) dw_s^r dt + E \int_0^T \zeta_t \int_0^t \int_Z (h_s^{(\varepsilon)}, \varphi) \tilde{\pi}(dz, ds) dt, \end{aligned} \quad (6.21)$$

where  $F$  and  $F_\varepsilon^i$ ,  $i = 1, 2, 3$ , are linear functionals of  $v \in \mathbb{V}_p^1$ , defined by

$$\begin{aligned} F(v) &= E \int_0^T \zeta_t (v_t, \varphi) dt, \quad F_\varepsilon^1 = E \int_0^T \zeta_t \int_0^t \langle \mathcal{A}_s^\varepsilon v_s, \varphi \rangle ds dt \\ F_\varepsilon^2(v) &= E \int_0^T \zeta_t \int_0^t (\mathcal{M}_s^{\varepsilon r} v_s, \varphi) dw_s^r dt \end{aligned}$$

and

$$F_\varepsilon^3(v) = E \int_0^T \zeta_t \int_0^t \int_Z (I^{\eta^{(\varepsilon)}} v_s + \gamma_{s,z}^{(\varepsilon)} T^{\eta^{(\varepsilon)}} v_s, \varphi) \tilde{\pi}(dz, ds) dt.$$

For each  $i = 1, 2, 3$  we also define the functional  $F^i$  in the same way as  $F_\varepsilon^i$  is defined above, but with  $\mathcal{A}, \mathcal{M}, \eta$  and  $\gamma$  in place of  $\mathcal{A}^\varepsilon, \mathcal{M}^\varepsilon, \eta^{(\varepsilon)}$  and  $\gamma^{(\varepsilon)}$  respectively. Obviously, by Hölder's inequality and the boundedness of  $\zeta$ , for all  $v \in \mathbb{V}_p^1$  we have

$$F(v) \leq C|v|_{\mathbb{V}_p^1}|\varphi|_{L_q}$$

with  $q = p/(p-1)$  and a constant  $C$  independent of  $v$  and  $\varepsilon$ , which means  $F \in \mathbb{V}_p^{1*}$ , the space of all bounded linear functionals on  $\mathbb{V}_p^1$ . Next we show that  $F_\varepsilon^i$  and  $F^i$  are also in  $\mathbb{V}_p^{1*}$  for each  $\varepsilon > 0$ , and  $F_\varepsilon^i \rightarrow F^i$  strongly in  $\mathbb{V}_p^{1*}$  as  $\varepsilon \rightarrow 0$  for  $i = 1, 2, 3$ .

**Lemma 6.3.** *For sufficiently small  $\varepsilon > 0$  the functionals  $F^i$  and  $F_\varepsilon^i$  are in  $\mathbb{V}_p^{1*}$  for  $i = 1, 2, 3$ .*

*Proof.* To show  $F_\varepsilon^1 \in \mathbb{V}_p^{1*}$  we notice that  $F_\varepsilon^1 = \sum_{k=1}^7 R_\varepsilon^k(v)$  with

$$\begin{aligned} R_\varepsilon^1(v) &= E \int_0^T \zeta_t \int_0^t -(a^{\varepsilon ij} D_j v_s, D_i \varphi) + (\bar{b}^{i(\varepsilon)} D_i v_s + c^{(\varepsilon)} v_s, \varphi) ds dt, \\ R_\varepsilon^2(v) &= E \int_0^T \zeta_t \int_0^t -(\mathcal{J}_{\xi^{(\varepsilon)}}^i v_s, D_i \varphi) ds dt, \quad R_\varepsilon^3(v) = E \int_0^T \zeta_t \int_0^t (\mathcal{J}_{\xi^{(\varepsilon)}}^0 v_s, \varphi) ds dt, \\ R_\varepsilon^4(v) &= E \int_0^T \zeta_t \int_0^t -(\mathcal{J}_{\eta^{(\varepsilon)}}^i v_s, D_i \varphi) ds dt, \quad R_\varepsilon^5(v) = E \int_0^T \zeta_t \int_0^t (\mathcal{J}_{\eta^{(\varepsilon)}}^0 v_s, \varphi) ds dt, \\ R_\varepsilon^6(v) &= E \int_0^T \zeta_t \int_0^t \int_Z ((\lambda^{\xi_{s,z}})^{(\varepsilon)} I^{\xi^{(\varepsilon)}} v_s, \varphi) \nu(dz) ds dt \\ R_\varepsilon^7(v) &= E \int_0^T \zeta_t \int_0^t \int_Z ((\lambda^{\eta_{s,z}})^{(\varepsilon)} I^{\eta^{(\varepsilon)}} v_s, \varphi) \mu(dz) ds dt \end{aligned} \quad (6.22)$$

where  $\mathcal{J}_{\xi^{(\varepsilon)}}^i, \mathcal{J}_{\eta^{(\varepsilon)}}^i, \mathcal{J}_{\xi^{(\varepsilon)}}^0, \mathcal{J}_{\eta^{(\varepsilon)}}^0$  are defined by (2.3) and (2.4) with  $\xi^{(\varepsilon)}$  and  $\eta^{(\varepsilon)}$  in place of  $\eta$  respectively, for  $i = 1, 2, \dots, d$ . Since the functions  $\zeta, a^\varepsilon, \bar{b}^{(\varepsilon)}$  and  $c^{(\varepsilon)}$  are in magnitude bounded by a constant, by Hölder's inequality we have

$$R_\varepsilon^i(v) \leq N|v|_{\mathbb{V}_p^1}|\varphi|_{H_q^1}, \quad (6.23)$$

for  $i = 1$  with  $q = p/(p-1)$  and a constant  $N$  independent of  $v$  and  $\varepsilon$ . which shows that  $R_\varepsilon^1 \in \mathbb{V}_p^{1*}$  for all  $\varepsilon$ . Using Taylor's formula

$$v(x + \theta \eta^{(\varepsilon)}) - v(x) = \int_0^1 D_i v(x + \vartheta \theta \eta^{(\varepsilon)}) \theta \eta^{(\varepsilon)i} d\vartheta,$$

and taking into account that  $|\zeta|$  is bounded by a constant, we have

$$R_\varepsilon^2(v) \leq C \int_0^1 \int_0^1 \int_0^T \int_Z \int_{\mathbb{R}^d} |Dv(s, x + \vartheta \theta \eta_{s,z}^{(\varepsilon)}(x))| \bar{\eta}^2(z) |D\varphi(x)| dx \mu(dz) ds d\theta d\vartheta.$$

Hence by Hölder's inequality and then the change of variable  $y = x + \vartheta \theta \eta_{s,z}^{(\varepsilon)}(x)$ , by Corollary 3.4 we get (6.23) for  $i = 2$  and  $\varepsilon \in (0, \varepsilon_0)$ , with  $\varepsilon_0$  given in Corollary 3.4, which proves that  $R_\varepsilon^2 \in \mathbb{V}_p^{1*}$  for  $\varepsilon \in (0, \varepsilon_0)$ . We can prove in the same way that  $R_\varepsilon^i \in \mathbb{V}_p^{1*}$  for  $i = 3, 4, 5$  and



$\varepsilon \in (0, \varepsilon_0)$ . Similarly, due to the boundedness of  $\zeta$ , Assumptions 2.2 and 2.4, using Lemma 3.1 and Hölder's inequality we have (6.23) for  $i = 6, 7$  for  $\varepsilon \in (0, \varepsilon_0)$ . Hence  $F_\varepsilon^1 \in \mathbb{V}_p^{1*}$  for  $\varepsilon \in (0, \varepsilon_0)$ . Due to the boundedness of  $\zeta$ ,  $\sigma^{(\varepsilon)r}$  and  $\beta^{(\varepsilon)r}$ , by Davis' and Hölder's inequalities, we get

$$F_\varepsilon^2(v) \leq CE \left( \int_0^T \sum_r |(\mathcal{M}_s^{\varepsilon r}, \varphi)|^2 ds \right)^{1/2} \leq C' |v|_{\mathbb{V}_p^1} |\varphi|_{L_q}$$

for  $v \in \mathbb{V}_p^1$  with constants  $C$  and  $C'$  independent of  $v$  and  $\varepsilon$ , which shows  $F_\varepsilon^2 \in \mathbb{V}_p^{1*}$ . By the boundedness of  $\zeta$ , using Davis' and Hölder's inequalities, we obtain

$$F_\varepsilon^3(v) \leq C(A_1(v) + A_2(v))$$

for  $v \in \mathbb{V}_p^1$  with a constant  $C$  independent of  $\varepsilon$  and  $v$ , where

$$A_1(v) = E \left( \int_0^T \int_Z |(I^{\eta^{(\varepsilon)}} v_s, \varphi)|^2 \mu(dz) ds \right)^{1/2},$$

and

$$A_2(v) = E \left( \int_0^T \int_Z |(\gamma_{s,z}^{(\varepsilon)} T^{\eta^{(\varepsilon)}} v_s, \varphi)|^2 \mu(dz) ds \right)^{1/2}.$$

Due to Assumptions 2.3 and 2.4 by Lemma 3.1 we have

$$A_i(v) \leq CE \left( \int_0^T \int_Z \bar{\eta}^2(z) |v_s|_{H_p^1}^2 |\varphi|_{L_q}^2 \mu(dz) ds \right)^{1/2} \leq C' |v|_{\mathbb{V}_p^1} |\varphi|_{L_q}$$

for  $i = 1, 2$  with constants  $C$  and  $C'$  independent of  $v$  and  $\varepsilon$ . Consequently,  $F_\varepsilon^3 \in \mathbb{V}_p^{1*}$ . In the same way we obtain  $F^i \in \mathbb{V}_p^{1*}$  for  $i = 1, 2, 3$ .  $\square$

**Lemma 6.4.** *For each  $i = 1, 2, 3$*

$$\lim_{\varepsilon \rightarrow 0} \sup_{|v|_{\mathbb{V}_p^1} \leq 1} |(F_\varepsilon^i - F^i)(v)| = 0. \quad (6.24)$$

*Proof.* We define the functionals  $R^i$  for  $i = 1, 2, \dots, 7$  in the same way as  $R_\varepsilon^i$  are defined in (6.22), but with  $a, b, c, \xi, \eta, \lambda^\xi$  and  $\lambda^\eta$  in place of  $a^\varepsilon, b^{(\varepsilon)}, c^{(\varepsilon)}, \xi^{(\varepsilon)}, \eta^{(\varepsilon)}, (\lambda^\xi)^{(\varepsilon)}$  and  $(\lambda^\eta)^{(\varepsilon)}$ , respectively. To prove  $F_\varepsilon^1 \rightarrow F^1$  strongly in  $\mathbb{V}_p^{1*}$ , we notice that

$$|F_\varepsilon^1(v) - F^1(v)| \leq \sum_{i=1}^7 |R_\varepsilon^i(v) - R^i(v)| \quad \text{for } v \in \mathbb{V}_p^1.$$

Since  $\zeta$  is bounded, for a constant  $N$  independent of  $v$  and  $\varepsilon$ , we have

$$|R_\varepsilon^1(v) - R^1(v)| \leq N \sum_{i=1}^3 Q_\varepsilon^i(v)$$

for all  $\varepsilon \geq 0$  with

$$Q_\varepsilon^1(v) := E \int_0^T \int_{\mathbb{R}^d} |D_j v(s, x)| |a^{\varepsilon ij}(s, x) - a^{ij}(s, x)| |D_i \varphi(x)| dx ds,$$

$$Q_\varepsilon^2(v) := E \int_0^T \int_{\mathbb{R}^d} |v(s, x)| |\bar{b}^{i(\varepsilon)}(s, x) - \bar{b}^i(s, x)| |D_i \varphi(x)| dx ds,$$

$$Q_\varepsilon^3(v) := E \int_0^T \int_{\mathbb{R}^d} |v(s, x)| |c^{(\varepsilon)}(s, x) - c(s, x)| |\varphi(x)| dx ds.$$

By Hölder's inequality and well-known properties of mollifications

$$\sup_{|v|_{\mathbb{V}_p^1} \leq 1} Q_\varepsilon^i(v) \leq N\varepsilon |\varphi|_{H_q^1} \quad \text{for } i = 1, 2, 3$$

with a constant  $N = N(K, d)$ , where  $q = p/(p-1)$ . Hence we get

$$\lim_{\varepsilon \rightarrow 0} \sup_{|v|_{\mathbb{V}_p^1} \leq 1} |(R_\varepsilon^i - R^i)(v)| = 0. \quad (6.25)$$

for  $i = 1$ . Clearly,

$$|R_\varepsilon^2(v) - R^2(v)| \leq H_\varepsilon^1(v) + H_\varepsilon^2(v) \quad (6.26)$$

with

$$\begin{aligned} H_\varepsilon^1(v) &:= \int_0^1 \int_0^T \int_Z (|\eta^{(\varepsilon)} - \eta| |v(s, \tau_{\theta\eta^{(\varepsilon)}}) - v(s)|, |D\varphi|) \mu(dz) ds d\theta, \\ H_\varepsilon^2(v) &:= \int_0^1 \int_0^T \int_Z (\bar{\eta} |v(s, \tau_{\theta\eta^{(\varepsilon)}}) - v(s, \tau_{\theta\eta})|, |D\varphi|) \mu(dz) ds d\theta. \end{aligned}$$

Note that  $|\eta^{(\varepsilon)} - \eta| \leq \varepsilon \bar{\eta}$  for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $z \in Z$  and  $\varepsilon > 0$ . Moreover, by Taylor's formula, Minkowski's inequality and Corollary 3.4

$$|v(s, \tau_{\theta\eta^{(\varepsilon)}}) - v(s)|_{L_p} \leq \bar{\eta} \int_0^1 |Dv(s, \tau_{\vartheta\theta\eta^{(\varepsilon)}})|_{L_p} d\vartheta \leq N\bar{\eta} |Dv(s)|_{L_p},$$

$$\begin{aligned} |v(s, \tau_{\theta\eta^{(\varepsilon)}}) - v(s, \tau_{\theta\eta})|_{L_p} &\leq \int_0^1 \|\eta^{(\varepsilon)} - \eta\| |Dv(s, (1-\vartheta)\tau_{\theta\eta^{(\varepsilon)}} + \vartheta\tau_{\theta\eta})|_{L_p} d\vartheta \\ &\leq N\varepsilon \bar{\eta} |Dv(s)|_{L_p} \end{aligned}$$

for  $s \in [0, T]$ ,  $z \in Z$ ,  $\omega \in \Omega$  and  $\varepsilon \in (0, \varepsilon_0)$ , with a constant  $N = N(K, d, p)$ . Hence by Hölder's inequality we have

$$H_\varepsilon^i \leq \varepsilon N |v|_{\mathbb{V}_p^1} |D\varphi|_{L_q} \int_Z \bar{\eta}^2(z) \mu(dz) = \varepsilon N K_\eta^2 |v|_{\mathbb{V}_p^1} |D\varphi|_{L_q} \quad \text{for } i = 1, 2 \text{ and } \varepsilon \in (0, \varepsilon_0),$$

which by virtue of (6.26) proves (6.25) for  $i = 2$ . We can prove similarly (6.25) for  $i = 3, 4, 5, 6, 7$ , which proves  $F_\varepsilon^1 \rightarrow F^1$  strongly. By the boundedness of  $\zeta$  and using Davis' and Hölder's inequalities we have

$$\begin{aligned} |F_\varepsilon^2(v) - F^2(v)| &\leq CE \left( \int_0^T |(\mathcal{M}_s^{\varepsilon r} v_s - \mathcal{M}_s^r v_s, \varphi) ds|^2 \right)^{1/2} \\ &\leq CE \left( \int_0^T \sum_{r=1}^\infty (|\sigma_s^{(\varepsilon)r} - \sigma_s^r| |Dv_s|, |\varphi|)^2 ds \right)^{1/2} \\ &\quad + CE \left( \int_0^T \sum_{r=1}^\infty (|\beta_s^{(\varepsilon)r} - \beta_s^r| |v_s|, |\varphi|)^2 ds \right)^{1/2} \\ &\leq C(A_\varepsilon^1(v) + A_\varepsilon^2(v)) \end{aligned}$$

for  $v \in \mathbb{V}_p^1$  and integers  $k \geq 1$  with a constant  $C$  independent of  $v$  and  $\varepsilon$ , where

$$A_\varepsilon^1(v) := E \left( \int_0^T |Dv_s|_{L_p}^2 \|\sigma_s^{(\varepsilon)} - \sigma_s\|_{L_q}^2 ds \right)^{1/2}$$

and

$$A_\varepsilon^2(v) := E \left( \int_0^T |v_s|_{L_p}^2 \|\beta_s^{(\varepsilon)} - \beta_s\|_{L_q}^2 ds \right)^{1/2}$$

with  $q = p/(p-1)$ . By standard properties of mollification

$$|\sigma_t^{(\varepsilon)} - \sigma_t| + |\beta_t^{(\varepsilon)} - \beta_t| \leq N\varepsilon$$

for all  $\varepsilon > 0$  and  $(x, t, \omega) \in H_T \times \Omega$  with a constant  $N = N(K, d)$ . Thus,

$$\sup_{|v|_{\mathbb{V}_p^1} \leq 1} A_\varepsilon^i(v) \leq \varepsilon N T^{(p-2)/2p} |\varphi|_{L_q} \quad \text{for } i = 1, 2$$

with  $q = p/(p-1)$  and a constant  $N = N(K, d)$ . Consequently, letting here  $\varepsilon \rightarrow 0$  we obtain (6.24) for  $i = 2$ . By the boundedness of  $\zeta$ , using Davis' inequality we get

$$|F_\varepsilon^3(v) - F^3(v)| \leq C(B_\varepsilon^1(v) + B_\varepsilon^2(v))$$

for  $v \in \mathbb{V}_p^1$  with a constant  $C$  independent of  $\varepsilon$  and  $v$ , where

$$B_\varepsilon^1(v) = E \left( \int_0^T \int_Z |(I^{\eta^{(\varepsilon)}} v_s - I^\eta v_s, \varphi)|^2 \mu(dz) ds \right)^{1/2}$$

and

$$B_\varepsilon^2(v) = E \left( \int_0^T \int_Z |(\gamma_{s,z}^{(\varepsilon)} T^{\eta^{(\varepsilon)}} v_s - \gamma_{s,z} T^\eta v_s, \varphi)|^2 \mu(dz) ds \right)^{1/2}.$$

Notice that by Taylor's formula

$$(I^{\eta^{(\varepsilon)}} v_s - I^\eta v_s, \varphi) = \int_{\mathbb{R}^d} \int_0^1 v_i(\chi_\theta^\varepsilon(s, z, x)) (\eta_{s,z}^{(\varepsilon)i}(x) - \eta_{s,z}^i(x)) \varphi(x) d\theta dx$$

where  $v_i := D_i v$  and

$$\chi_\theta^\varepsilon(s, z, x) := x + \theta \eta_{s,z}^{(\varepsilon)}(x) + (1 - \theta) \eta_{s,z}(x)$$

for  $\theta \in (0, 1)$ ,  $\varepsilon > 0$  and  $(s, z, \omega) \in [0, T] \times Z \times \Omega$ . By Corollary 3.4 there are positive constants  $\varepsilon_0$  and  $M = M(K, d, m)$  such that the function  $\chi_\theta^\varepsilon(s, z, \cdot)$  is a  $C^{\bar{m}}$ -diffeomorphism on  $\mathbb{R}^d$  for every  $\varepsilon \in (0, \varepsilon_0)$ ,  $\theta \in (0, 1)$ ,  $\omega \in \Omega$ ,  $t \in [0, T]$   $z \in Z$ , and

$$|D\chi_\theta^\varepsilon(s, z, x)| \leq M \quad \text{and} \quad M^{-1} \leq |\det D\chi_\theta^\varepsilon(s, z, x)|$$

for all  $(x, s, z, \omega) \in H_T \times Z \times \Omega$ . Due to Assumption 2.3 we have

$$|\eta_{s,z}^{(\varepsilon)}(x) - \eta_{s,z}(x)| \leq \varepsilon \bar{\eta}(z) \quad \text{for all } \varepsilon > 0 \text{ and } (s, x, \omega, z) \in H_T \times \Omega \times Z.$$

Thus using Hölder's inequality we get

$$\begin{aligned} B_\varepsilon^1(v) &\leq E \left( \int_0^T \int_0^1 \int_Z |Dv(\chi_\theta^\varepsilon(s, z))|_{L_p}^2 |\varphi|_{L_q}^2 \varepsilon^2 \bar{\eta}^2(z) \mu(dz) d\theta ds \right)^{1/2} \\ &\leq C \varepsilon |\varphi|_{L_q} |v|_{\mathbb{V}_p^1} |\bar{\eta}|_{L_2} \end{aligned} \tag{6.27}$$

with a constant  $C$  independent of  $\varepsilon$  and  $v$ . Furthermore we notice that by Assumption 2.4

$$|\gamma_{s,z}^{(\varepsilon)}(x) - \gamma_{s,z}(x)| \leq N \varepsilon \bar{\eta}(z)$$

with a constant  $N = N(K, d)$  for all  $\varepsilon > 0$  and  $(s, z, \omega, z) \in H_T \times \Omega \times Z$ . Hence, in a similar way as the estimate of  $B_\varepsilon^1(v)$  is obtained, we can show

$$B_\varepsilon^2(v) \leq C\varepsilon|\varphi|_{L_q}|v|_{\mathbb{V}_p^1} \quad (6.28)$$

with a constant  $C$  independent of  $\varepsilon$  and  $v$ . Hence, combining (6.27) and (6.28) we get (6.24) for  $i = 3$ .  $\square$

Since  $F_\varepsilon^i \rightarrow F^i$  strongly in  $\mathbb{V}_p^{1*}$  as  $\varepsilon \rightarrow 0$  and  $u^{\varepsilon_k} \rightarrow u$  weakly in  $\mathbb{V}_p^1$  as  $\varepsilon_k \rightarrow 0$ , we have

$$\lim_{k \rightarrow \infty} F(u^{\varepsilon_k}) = F(u), \quad \lim_{k \rightarrow \infty} F_k^i(u^{\varepsilon_k}) = F^i(u) \quad \text{for } i = 1, 2, 3.$$

By well-known properties of mollifications and using Lemma 3.5 it is easy to show

$$\begin{aligned} \lim_{k \rightarrow \infty} E \int_0^T \zeta_t(\psi^{(\varepsilon_k)}, \varphi) dt &= E \int_0^T \zeta_t(\psi, \varphi) dt, \\ \lim_{k \rightarrow \infty} E \int_0^T \int_0^t \zeta_t(f_s^{(\varepsilon_k)}, \varphi) ds dt &= E \int_0^T \int_0^t \zeta_t(f_s, \varphi) ds dt, \\ \lim_{k \rightarrow \infty} E \int_0^T \zeta_t \int_0^t (g_s^{(\varepsilon_k)r}, \varphi) dw_s^r dt &= E \int_0^T \zeta_t \int_0^t (g_s^r, \varphi) dw_s^r dt, \end{aligned}$$

and

$$\lim_{k \rightarrow \infty} E \int_0^T \zeta_t \int_0^t \int_Z (h_s^{(\varepsilon_k)}, \varphi) \tilde{\pi}(dz, ds) dt = E \int_0^T \zeta_t \int_0^t \int_Z (h_s, \varphi) \tilde{\pi}(dz, ds) dt.$$

Hence, taking  $k \rightarrow \infty$  in equation (6.21) we get

$$\begin{aligned} E \int_0^T \zeta_t(u_t, \varphi) dt &= E \int_0^T \zeta_t(\psi, \varphi) dt + E \int_0^T \zeta_t \int_0^t \langle \mathcal{A}u_s, \varphi \rangle ds dt \\ &\quad + E \int_0^T \zeta_t \int_0^t (f_s, \varphi) ds dt + E \int_0^T \zeta_t \int_0^t (\mathcal{M}_s^r u_s + g_s^r, \varphi) ds dt \\ &\quad + E \int_0^T \zeta_t \int_0^t \int_Z (I^\eta u_s + \gamma_{s,z} T^\eta u_s + h_s, \varphi) \tilde{\pi}(dz, ds) dt \end{aligned}$$

for every bounded well-measurable process  $\zeta$  and every  $\varphi \in C_0^\infty$ , which implies that for every  $\varphi \in C_0^\infty$  equation (1.1) holds  $P \otimes dt$  almost everywhere. Hence, by Lemma 3.10  $u$  has an  $L_p$ -valued cadlag modification, denoted also by  $u$ , which is a generalised solution to (1.1)-(1.2). Moreover, from (6.19) we obtain

$$|u|_{\mathbb{V}_{r,p}^n} \leq \liminf_{\varepsilon_k \rightarrow 0} |u^{\varepsilon_k}|_{\mathbb{V}_{r,p}^n} \leq N(|\psi|_{\Psi_p^n} + |f|_{\mathbb{H}_p^n} + |g|_{\mathbb{H}_p^{n+1}(l_2)} + |h|_{\mathbb{H}_p^{n+i}(\mathcal{L}_{p,2})})$$

for  $n = 0, 1, \dots, m$  for every integer  $r > 1$  with a constant  $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ , where  $i = 1$  when  $p = 2$  and  $i = 2$  for  $p > 2$ . Letting here  $r \rightarrow \infty$  we obtain

$$E \operatorname{ess\,sup}_{t \in [0, T]} |u_t|_{H_p^n}^p \leq N(E|\psi|_{H_p^n}^p + EK_{n,p}^p(T)) \quad (6.29)$$

for  $n = 0, 1, 2, \dots, m$  with a constant  $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ . We already know that  $u$  is and  $L_p$ -valued cadlag process. Hence, applying Lemma 3.12 with  $V = H_p^m$ ,  $U = H_p^0$  and

we obtain that  $u$  is weakly cadlag as an  $H_p^m$ -valued process, and we can change the essential supremum into supremum in (6.29), i.e.,

$$E \sup_{t \in [0, T]} |u_t|_{H_p^n}^p \leq N(E|\psi|_{H_p^n}^p + EK_{n,p}^p(T)) \quad \text{for } n = 0, 1, 2, \dots, m. \quad (6.30)$$

Thus we can also see that  $u$  is strongly cadlag as an  $H_p^{m-1}$ -valued process. To dispense with Assumption 6.1 we take a non-negative function  $\chi \in C_0^\infty(\mathbb{R}^d)$  such that  $\chi(x) = 1$  for  $|x| \leq 1$  and  $\chi(x) = 0$  for  $|x| \geq 2$ , and for integers  $n \geq 1$  define

$$\begin{aligned} \psi^n(x) &= \psi(x)\chi_n(x), & f_t^n(x, z) &= f_t(x, z)\chi_n(x), \\ g_t^{nr}(x) &= g_t^r(x)\chi_n(x), & h_t^n(x, z) &= h_t(x, z)\chi_n(x) \end{aligned}$$

for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $z \in Z$ , where  $\chi_n(x) = \chi(x/n)$ . Then for each  $n$  there is a unique generalised solution  $u^n = (u_t^n)_{t \in [0, T]}$  to equation (1.1)-(1.2) with  $\psi^n$ ,  $f^n$ ,  $g^n$  and  $h^n$  in place of  $\psi$ ,  $f$ ,  $g$  and  $h$ , respectively. Moreover by (6.30)

$$\begin{aligned} E \sup_{t \leq T} |u_t^n - u_t^l|_{H_p^m}^p &\leq NE|\psi^n - \psi^l|_{H_p^m}^p + NE \int_0^T |f_s^n - f_s^l|_{H_p^m}^p ds \\ &+ NE \int_0^T (|g_s^n - g_s^l|_{H_p^{m+1}(l_2)}^p + |h_s^n - h_s^l|_{H_p^{m+1}(\mathcal{L}_{p,2})}^p + \mathbf{1}_{p \geq 2} |h_s^n - h_s^l|_{H_p^{m+2}(\mathcal{L}_{p,2})}^p) ds \end{aligned}$$

with a constant  $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ . Letting here  $l, n \rightarrow \infty$  we get

$$\lim_{n, l \rightarrow \infty} E \sup_{t \leq T} |u_t^n - u_t^l|_{H_p^m}^p = 0.$$

Consequently, there is an  $H_p^m$ -valued adapted process  $u = (u_t)_{t \in [0, T]}$  such that for a subsequence  $n' \rightarrow \infty$  we have  $\sup_{t \leq T} |u_t^{n'} - u_t|_{H_p^m} \rightarrow 0$  almost surely. Hence  $u$  is an  $H_p^{m-1}$ -valued cadlag process, and it is easy to show that it is a generalised solution to (1.1)-(1.2), such that (6.30) holds and  $u$  is weakly cadlag as an  $H_p^m$ -valued process.

The next theorem extends the above result to equation (1.1)-(1.2) with  $\mathcal{R}$  satisfying Assumption 2.5.

**Theorem 6.5.** *Let Assumptions 2.1 through 2.6 hold with an integer  $m \geq 1$  and a real number  $p \geq 2$ . Assume also (6.16). Then equation (1.1)-(1.2) has a unique generalised solution  $u$ , such that  $u$  is an  $H_p^m$ -valued weakly cadlag process, satisfying estimate (6.30), and it is cadlag as an  $H_p^{m-1}$ -valued process.*

*Proof.* We use the standard method of continuity, see, e.g., [23]. For  $\lambda \in [0, 1]$ , we consider the equation

$$\begin{aligned} du_t &= (\mathcal{A}_t^0 u_t + \lambda \mathcal{R}_t u_{t-} + f_t) dt + (\mathcal{M}_t^r u_t + g_t^r) dw_t^r \\ &+ \int_Z (I^\eta u_{t-} + \gamma_t I^\eta u_{t-} + h_t) \tilde{\pi}(dz, dt) \end{aligned} \quad (6.31)$$

for  $(t, x) \in H_T$  with initial condition

$$u_0 = \psi, \quad (6.32)$$

where  $\mathcal{A}_t^0 = \mathcal{L}_t + \mathcal{N}_t^\xi + \mathcal{N}_t^\eta$  for every  $t \in [0, T]$ . We look for a solution  $u$  from the space  $\mathcal{H}_p^m$  of  $\mathcal{F}_t$ -adapted  $H_p^m$ -valued weakly cadlag processes which are strongly cadlag as  $H_p^{m-1}$ -valued processes such that  $|u|_{\mathcal{H}_p^m}^p = E \sup_{t \leq T} |u_t|_{H_p^m}^p < \infty$ . Notice that  $\mathcal{H}_p^m$  is a Banach space. If

$u \in \mathcal{H}_p^m$  is a generalised solution to (6.31), then by Assumption 2.5 and estimate (6.30) we have

$$\begin{aligned} E \sup_{s \leq t} |u_s|_{H_p^n}^p &\leq NE|\psi|_{H_p^n}^p + NE \int_0^t |\lambda \mathcal{R}_s u_{s-}|_{H_p^n}^p ds + NEK_{p,n}^p(T) \\ &\leq NE|\psi|_{H_p^n}^p + NK \int_0^t E \sup_{r \leq s} |u_r|_{H_p^n}^p ds \quad \text{for } n = 0, 1, \dots, m \end{aligned}$$

with a constant  $N = N(m, d, p, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ . Hence by Gronwall's lemma we have estimate (6.30) for  $u$ . Let  $\Lambda$  denote the set of  $\lambda \in [0, 1]$  such that for any  $\psi \in \Psi_p^m = L_p(\Omega, \mathcal{F}_0; H_p^m)$ ,  $f \in \mathbb{H}_p^m$ ,  $g \in \mathbb{H}_p^{m+1}(l^2)$  and  $h \in \mathbb{H}_p^{m+i}(\mathcal{L}_{p,2})$ , with  $i = 1$  when  $p = 2$  and  $i = 2$  when  $p > 2$ , equation (6.31)-(6.32) has a unique generalised solution in  $\mathcal{H}_p^m$ . Clearly  $0 \in \Lambda$ , and we need to prove  $1 \in \Lambda$ . To this end, it suffices to show that there is an  $\delta > 0$  such that for any  $\lambda_0 \in \Lambda$ ,

$$[\lambda_0 - \delta, \lambda_0 + \delta] \cap [0, 1] \in \Lambda.$$

Fix  $\lambda_0 \in \Lambda$ ,  $\psi \in \Psi_p^m$ ,  $f \in \mathbb{H}_p^m$ ,  $g \in \mathbb{H}_p^{m+1}(l^2)$  and  $h \in \mathbb{H}_p^{m+i}(\mathcal{L}_{p,2})$ . For  $v \in \mathcal{H}_p^m$  and  $\lambda \in [0, 1]$  we consider the equation

$$\begin{aligned} du_t &= (\mathcal{A}_t^0 u_t + \lambda_0 \mathcal{R}_t u_{t-} + (\lambda - \lambda_0) \mathcal{R}_t v_{t-} + f_t) dt + (\mathcal{M}_t^r u_t + g_t^r) dw_t^r \\ &\quad + \int_Z (I^\eta u_{t-} + \gamma_t T^\eta u_{t-} + h_t) \tilde{\pi}(dz, dt) \end{aligned}$$

for  $(t, x) \in H_T$ , with initial condition  $u_0 = \psi$ . Since  $\lambda_0 \in \Lambda$ , this problem has a unique generalised solution  $u \in \mathcal{H}_p^m$ . Define the operator  $Q_\lambda$  by  $u = Q_\lambda v$ . Then  $Q_\lambda$  maps  $\mathcal{H}_p^m$  into itself, and  $\lambda \in \Lambda$  if and only if there is a fixed point of  $Q_\lambda$ . If  $v^i \in \mathcal{H}_p^m$  and  $u^i = Q_\lambda v^i$  for  $i = 1, 2$ , then for  $u := u^2 - u^1$  we have

$$\begin{aligned} du_t &= (\mathcal{A}_t^0 u_t + \lambda_0 \mathcal{R}_t u_{t-} + (\lambda - \lambda_0) \mathcal{R}_t (v_t^2 - v_t^1)) dt + \mathcal{M}_t^r u_t dw_t^r \\ &\quad + \int_Z (I^\eta u_{t-} + \gamma_t T^\eta u_{t-}) \tilde{\pi}(dz, dt), \quad (t, x) \in H_T, \end{aligned}$$

with  $u_0 = 0$ . Hence, using estimate (6.30) for  $u$ , due to Assumption 2.5 on  $\mathcal{R}$  we get

$$|Q_\lambda v^2 - Q_\lambda v^1|_{\mathcal{H}_p^m} \leq N|\lambda - \lambda_0| |\mathcal{R}(v^2 - v^1)|_{\mathcal{H}_p^m} \leq N'|\lambda - \lambda_0| |v^2 - v^1|_{\mathcal{H}_p^m}$$

with constants  $N$  and  $N'$  depending only on  $m, d, p, T, K, K_{\bar{\xi}}$  and  $K_{\bar{\eta}}$ . Taking  $\delta = (2N')^{-1}$  we obtain that  $Q_\lambda$  is a contraction mapping on  $\mathcal{H}_p^m$  if  $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta] \cap [0, 1]$ . Consequently, (6.31) has a unique solution  $u$  in  $\mathcal{H}_p^m$ , and it satisfies (6.30).  $\square$

If  $m \geq 1$  is not an integer, then we set  $\theta = m - [m]$  and by Theorem 4.1 we have

$$[\Psi_p^{[m]}, \Psi_p^{[m]}]_\theta = \Psi_p^m, \quad [\mathbb{H}_p^{[m]}, \mathbb{H}_p^{[m]}]_\theta = \mathbb{H}_p^m,$$

$$[\mathbb{H}_p^{[m]+1}(l_2), \mathbb{H}_p^{[m]+1}(l_2)]_\theta = \mathbb{H}_p^{m+1}(l_2), \quad [\mathbb{H}_p^{[m]+i}(\mathcal{L}_{p,2}), \mathbb{H}_p^{[m]+i}(\mathcal{L}_{p,2})]_\theta = \mathbb{H}_p^{m+i}(\mathcal{L}_{p,2})$$

for  $i = 1, 2$ , and

$$\mathbb{U}_{r,p}^m = [\mathbb{U}_{r,p}^{[m]}, \mathbb{U}_{r,p}^{[m]}]_\theta$$

for integers  $r > 1$ . If Assumptions 2.1 through 2.6 with  $m \geq 1$  hold then we have shown above that the solution operator  $\mathbb{S}$ , which maps the data  $(\psi, f, g, h)$  into the generalised solution  $u$  of (1.1)-(1.2), is continuous from

$$\Psi_p^{[m]} \times \mathbb{H}_p^{[m]} \times \mathbb{H}_p^{[m]+1}(l_2) \times \mathbb{H}_{p,2}^{[m]+i}$$

to  $\mathbb{U}_{p,r}^{[m]}$ , and from

$$\Psi_p^{[m]} \times \mathbb{H}_p^{[m]} \times \mathbb{H}_p^{[m]+1}(l_2) \times \mathbb{H}_{p,2}^{[m]+i}$$

to  $\mathbb{U}_{p,r}^{[m]}$ , for  $i = 1$  when  $p = 2$  and for  $i = 2$  when  $p > 2$ , with operator norms bounded by a constant  $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$ . Hence by Theorem 4.1 (i) we have

$$|u|_{\mathbb{U}_{r,p}^p}^p \leq N(E|\psi|_{H_p^m}^p + EK_{m,p}^p(T))$$

with a constant  $N = N(p, d, m, T, K, K_{\bar{\eta}})$ . In the same way we get

$$|u|_{\mathbb{U}_{r,p}^s}^p \leq N(E|\psi|_{H_p^s}^p + EK_{s,p}^p(T)) \quad \text{for any } s \in [0, m].$$

Now, like before, letting here  $r \rightarrow \infty$  we obtain (6.29) for real numbers  $s \in [0, m]$ , and using Lemma 3.12 we get that  $u$  is an  $H_p^m$ -valued weakly cadlag process such that (6.30) holds for any  $s \in [0, m]$ . Taking into account that  $u$  is a strongly cadlag  $L_p$ -valued process and using the interpolation inequality Theorem 4.1(v) with  $A_0 := L_p$  and  $A_1 := H_p^m$ , we get that  $u$  is strongly cadlag as an  $H_p^s$ -valued process for every real number  $s < m$ .

Finally we can prove estimate (2.6) for  $q \in (0, p)$  by applying Lemma 3.11 in the same way as it is used in [16] to prove the corresponding supremum estimate therein.

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